PROJECTION ALGORITHMS: RESULTS AND OPEN PROBLEMS

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In this note, I review basic results and open problems in the area of projection algorithms. My aim is to generate interest in this fascinating field, and to highlight the fundamental importance of bounded linear regularity.
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1. INTRODUCTION

We assume throughout that

$$\boxed{X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle \text{ and induced norm } \|\cdot\|.}$$

The first general projection algorithm — the *method of alternating projections* — was studied by John von Neumann in 1933:

Fact 1.1 (von Neumann). [38] Suppose C_1, C_2 are two closed subspaces in X with corresponding projections P_1, P_2. Let $C := C_1 \cap C_2$ and fix a starting point $x_0 \in X$. Then the *sequence of alternating projections* generated by

$$x_1 := P_1x_0, x_2 := P_2x_1, x_3 := P_1x_2, \ldots$$

converges in norm to the projection of x_0 onto C.

In view of its conceptual simplicity and elegance, it is not surprising that Fact 1.1 has been generalized and rediscovered many times. (See the Deutsch's [26,25] for further information. Other algorithmic approaches are possible via generalized inverses [2].)

In this note, I consider some of the many generalizations of von Neumann's result, and discuss the intriguing open problems these generalizations spawned. My aim is to demonstrate that bounded linear regularity*, a quantitative geometric property of a collection of sets, is immensely useful and plays a crucial role in several results related to the open problems.*

The material is organized as follows.

In Section 2, we review basic convergence results by Halperin, by Bregman, and by Gubin, Polyak, and Raik. After recalling helpful properties of Fejér monotone sequences and of boundedly linearly regular collections of sets, we show how these notions work together in the proof a prototypical convergence result (Theorem 2.10).

Bounded linear regularity is reviewed in Section 3. Metric regularity, a notion ubiquitous in optimization, is shown to be genuinely stronger than bounded linear regularity. We also mention the beautiful relationship to conical open mapping theorems.

In the remaining sections, we discuss old and new open problems related to the inconsistent case (Section 4), to random projections (Section 5), and to acceleration (Section 6).

2. WEAK VS NORM VS LINEAR CONVERGENCE

It is very natural to try to generalize Fact 1.1 from two to finitely many subspaces. Israel Halperin achieved this, with a proof very different from von Neumann's, in 1962.

Fact 2.1 (Halperin). [31] Suppose $C_1, \dots, C_N$ are finitely many closed subspaces in X with corresponding projections $P_1, \dots, P_N$. If $C := \bigcap_{i=1}^N C_i$ and $x_0 \in X$, then the *sequence of cyclic projections*

$$x_1 := P_1 x_0, x_2 := P_2 x_1, \dots, x_N := P_N x_{N-1}, x_{N+1} := P_1 x_N, \dots$$

converges *in norm* to the projection of x_0 onto C.

See also Bruck and Reich's [18] and Baillon, Bruck, and Reich's [4] for various extensions of Halperin's result to uniformly convex Banach spaces and more general (not necessarily linear) mappings.

We assume from now on that

$$\boxed{C_1, \dots, C_N \text{ are finitely many } (N \geq 2) \text{ closed convex sets with projections } P_1, \dots, P_N,}$$

and

$$\boxed{C := \bigcap_{i=1}^N C_i.}$$

For the reader's convenience, we recall some basic properties of projections.

Fact 2.2. Suppose S is a closed convex nonempty set in X, and $x \in X$. Then there exists a unique point in S, denoted $P_S x$ and called the *projection of x onto S*, with $\|x - P_S x\| = \min_{s \in S} \|x - s\| =: d(x, S)$. This point is characterized by

$$P_S x \in S \text{ and } \langle S - P_S x, x - P_S x \rangle \leq 0.$$

Moreover, P_S is *firmly nonexpansive*:

$$\|P_S x - P_S y\|^2 + \|(x - P_S x) - (y - P_S y)\|^2 \leq \|x - y\|^2;$$

in particular, P_S is *nonexpansive*:

$$\|P_S x - P_S y\| \leq \|x - y\|, \quad \forall x \in X, \forall y \in X.$$

What happens if we drop the assumption on linearity of the sets in Fact 2.1? In 1965, Lev Bregman proved the following fundamental result on a projection algorithm with general convex sets:

Fact 2.3 (Bregman). [17] If $C \neq \emptyset$, then the sequence of cyclic projections converges *weakly* to some point in C.

Remark 2.4. Several comments are in order.

(i) Many optimization problems can be recast as a *Convex Feasibility Problem*:

$$\text{find } x \in C.$$

For instance, *Linear Programming* has such a reformulation (in terms of primal feasibility, dual feasibility, and complementary slackness). Thus Bregman's result opened the door to solve important classes of convex optimization problems by using projection algorithms; see Censor and Zenios's recent monograph [19].

(ii) In contrast to Halperin's result, Bregman obtained weak convergence only.

(iii) Halperin's result identifies the limit of the sequence of cyclic projections as the point in C that is nearest to the starting point. Simple examples (consider two distinct but intersecting halfspaces in $\mathbb{R}^2$) show that cyclic projections may fail to converge to the nearest point in C.

Because of Remark 2.4.(ii), the following problem is of outstanding importance:

Open Problem 1. In Bregman's result (Fact 2.3), can the convergence be only weak?

Remark 2.5. During the Haifa workshop on inherently parallel algorithms, Hein Hundal announced the existence of two closed convex intersecting sets in ℓ_2 and a particular starting point where the method of alternating projections converges weakly but not in norm. (Full details were not available at the time of the writing of this note.)

However, it will always be interesting to know when the convergence is *linear*. Two disks in $\mathbb{R}^2$ that touch in exactly one point demonstrate that *something* is needed in order to guarantee linear convergence. This is best achieved by imposing bounded linear regularity:

Definition 2.6. [10] Suppose $C \neq \emptyset$. Then the collection $\{C_1, \ldots, C_N\}$ is *boundedly linearly regular*, if for every bounded set S in X, there exists $\kappa > 0$ such that

$$d(s, C) \leq \kappa \max\{d(s, C_1), \ldots, d(s, C_N)\}, \quad \forall s \in S.$$

If we can find $\kappa > 0$ so that the last inequality is true for every $s \in X$, then $\{C_1, \ldots, C_N\}$ is *linearly regular*. Finally, $\{C_1, \ldots, C_N\}$ is *boundedly regular*, if $d(y_n, C) \to 0$, for every bounded sequence (y_n) with $\max_{1 \leq i \leq N} d(y_n, C_i) \to 0$.

Clearly, linear regularity $\Rightarrow$ bounded linear regularity $\Rightarrow$ bounded regularity.

Gubin, Polyak, and Raik implicitly proved the next result in their fundamental paper from 1965:

Fact 2.7 (Gubin, Polyak, and Raik). [30] If $C \neq \emptyset$ and $\{C_1, \dots, C_N\}$ is boundedly linearly regular, then the sequence of cyclic projections converges *linearly* to some point in C.

We now recall the definition of Fejér monotonicity.

Definition 2.8. Suppose S is a closed convex nonempty set in X, and $(y_n)_{n\geq 0}$ is a sequence in X. Then (y_n) is *Fejér monotone with respect to* S, if

$$\|y_{n+1} - s\| \leq \|y_n - s\|, \quad \forall n \geq 0, \forall s \in S.$$

Fejér monotone sequence have various pleasant properties; see [22,23], [10], and [6]. Here, we focus on characterizations of convergence, which will come handy when studying algorithms:

Fact 2.9. Suppose $(y_n)_{n\geq 0}$ is Fejér monotone with respect to a closed convex nonempty set S in X. Then:

(i) (y_n) is bounded, and $(d(y_n, S))$ is decreasing.

(ii) $P_S(y_n)$ converges in norm to some point $\bar{s} \in S$.

(iii) (y_n) converges weakly to $\bar{s} \Leftrightarrow$ all weak cluster points of (y_n) lie in S.

(iv) (y_n) converges in norm to $\bar{s} \Leftrightarrow d(y_n, S) \to 0$.

(v) (y_n) converges linearly to $\bar{s} \Leftrightarrow \exists\, \theta \in [0, 1)$ with $d(y_{n+1}, S) \leq \theta d(y_n, S)$, $\forall n \geq 0$.

It is highly instructive to see the general structure of the proofs of Fact 2.3 and Fact 2.7. For clarity, we consider only alternating projections.

Theorem 2.10. (Prototypical Convergence Result) Suppose $N = 2$, $C = C_1 \cap C_2 \neq \emptyset$, and $(x_n)_{n\geq 0}$ is a sequence of alternating projections. Then (x_n) is Fejér monotone with respect to C, and

$$\max\big\{d^2(x_n, C_1), d^2(x_n, C_2)\big\} \leq d^2(x_n, C) - d^2(x_{n+1}, C), \quad \forall n \geq 0. \tag{$*$}$$

Let $\bar{c} = \lim_n P_C(x_n)$. Then:

(i) (x_n) *always* converges *weakly* to $\bar{c}$.

(ii) If $\{C_1, C_2\}$ is *boundedly regular*, then (x_n) converges *in norm* to $\bar{c}$.

(iii) If $\{C_1, C_2\}$ is *boundedly linearly regular*, then (x_n) converges *linearly* to $\bar{c}$.

(iv) If $\{C_1, C_2\}$ is *linearly regular*, then (x_n) converges *linearly* to $\bar{c}$ with a *rate independent of the starting point.*

Proof. Using (firm) nonexpansiveness of projections (Fact 2.2), we obtain easily $(*)$ and Fejér monotonicity of (x_n). Now the right-hand side of $(*)$ tends to 0 (use Fact 2.9.(i)); hence, so does the left-hand side of $(*)$, and its square root:

$$\max\{d(x_n, C_1), d(x_n, C_2)\} \to 0. \tag{$**$}$$

(i): Suppose x is an arbitrary weak cluster point of (x_n). Use $(**)$ and the weak lower semicontinuity of $d(\cdot, C_1), d(\cdot, C_2)$ to conclude that $d(x, C_1) = d(x, C_2) = 0$. Thus $x \in C = C_1 \cap C_2$, and we are done by Fact 2.9.(iii).

(ii): By $(**)$ and bounded regularity, $d(x_n, C) \to 0$. Apply Fact 2.9.(iv).

(iii): The set $\{x_n : n \geq 0\}$ is bounded (Fact 2.9.(i)). Bounded linear regularity yields $\kappa > 0$ such that

$$d(x_n, C) \leq \kappa \max\{d(x_n, C_1), d(x_n, C_2)\}, \quad \forall n \geq 0.$$

Square this, and combine with "κ^2 times $(*)$" to get

$$d^2(x_n, C) \leq \kappa^2 \big(d^2(x_n, C) - d^2(x_{n+1}, C)\big);$$

equivalently, after re-arranging, $d(x_{n+1}, C) \leq \sqrt{1 - 1/\kappa^2}\, d(x_n, C)$. (Note that κ must be greater than or equal to 1 — because C is a subset of both C_1 and C_2 — and so the radical is always nonnegative.) The result follows now from Fact 2.9.(v).

(iv): is analogous to (iii). ☐

Remark 2.11. A second look at the proof of Theorem 2.10 reveals that only Fejér monotonicity and $(*)$ is required to arrive at the conclusions.

We conclude this section by noting that all results have analogues for much broader classes of algorithms, including various *parallel schemes*. As a starting point to the vast literature, we refer the reader to [21,20], [34], [35], [36], [37], and [6].

3. BOUNDED LINEAR REGULARITY

We now collect the facts on bounded linear regularity that are most relevant to us.

Fact 3.1. Suppose $N = 2$ and $C = C_1 \cap C_2 \neq \emptyset$. Then $\{C_1, C_2\}$ is boundedly linearly regular whenever one of the following conditions holds.

(i) $C_1 \cap \operatorname{int} C_2 \neq \emptyset$;

(ii) $0 \in \operatorname{int}(C_2 - C_1)$;

(iii) C_1, C_2 are subspaces and $C_1 + C_2$ is closed;

(iv) $\{r(c_2 - c_1) : r \geq 0, c_1 \in C_1, c_2 \in C_2\}$ is a closed subspace.

Proof. (i): [30]. (ii), (iii), and (iv): [8]. ☐

Condition (iv) of Fact 3.1 subsumes, in fact, conditions (i)—(iii). We now turn to the general case.

Fact 3.2. Suppose $N \geq 2$ and $C = \bigcap_{i=1}^{N} C_i \neq \emptyset$. Then $\{C_1, \ldots, C_N\}$ is boundedly linearly regular whenever one of the following conditions holds.

(i) *reduction to two sets*: each $\{C_1 \cap \cdots C_i, C_{i+1}\}$ is boundedly linearly regular;

(ii) *standard constraint qualification*: $X = \mathbb{R}^M$, $\bigcap_{i=1}^{r} \mathrm{ri}(C_i) \cap \bigcap_{i=r+1}^{N} C_i \neq \emptyset$, and the sets $C_{r+1}, \ldots, C_N$ are polyhedral, for some $0 \leq r \leq N$;

(iii) *for subspaces*: each C_i is a subspace, and $C_1^{\perp} + \cdots + C_N^{\perp}$ is closed;

(iv) *Hoffman's inequality*: each C_i is a polyhedron.

Proof. (i): [10, Theorem 5.11]. (ii): [6, Theorem 5.6.2] or [12, Corollary 5]. (iii): [10, Theorem 5.19]. (iv): essentially [32]; see also [6, Corollary 5.7.2]. $\square$

Remark 3.3. • Combining Fact 3.1 and Fact 3.2 with Fact 2.7 yields a number of classical results on cyclic projections. (See [6, Section 9.5].) • However, this approach is incapable of recovering the basic results by von Neumann (Fact 1.1) and by Halperin (Fact 2.1). (Those results are best derived via *Dykstra's algorithm*; see [16] for this beautiful and powerful method. In contrast to the method of cyclic projections (Remark 2.4.(iii)), Dykstra's algorithm yields the projection of the starting point onto C and thus a well-recognized and most useful limit.)

Fact 3.4. [8, Theorem 5.3.(iv)] In $X := \ell_2$, let C_1 be the cone of nonnegative sequences and C_2 be an arbitrary hyperplane with $C = C_1 \cap C_2 \neq \emptyset$. Then the sequence of alternating projections always converges *in norm*.

The proof of Fact 3.4 considers two cases: presence and absence of bounded (linear) regularity. In the latter case, an order-theoretic argument yields norm, but not linear, convergence. It would be interesting to find out how to tackle the following, closely related problem:

Open Problem 2. What happens in Fact 3.4 if replace C_2 by an affine subspace of finite codimension?

Bounded linear regularity vs metric regularity

The notion of metric regularity of a set-valued map is well-known. How does it compare to bounded linear regularity? Metric regularity turns out to be a much stronger property.

Definition 3.5. Suppose $C = \bigcap_{i=1}^{N} C_i \neq \emptyset$. Then $\{C_1, \ldots, C_N\}$ is *metrically regular* if for every $c \in C$, there exists $K > 0$ such that for every $x \in X$ sufficiently close to c and for all $(b_1, \ldots, b_N) \in X^N$ sufficiently close to $0 \in X^N$, we have

$$d\big(x, (C_1 - b_1) \cap \cdots \cap (C_N - b_N)\big) \leq K\big(d(x + b_1, C_1) + \cdots + d(x + b_N, C_N)\big).$$

Remark 3.6. In the language of set-valued analysis, Definition 3.5 precisely states that the set-valued map $\Omega(x) := (C_1 \times \cdots \times C_N) - (x, \ldots, x)$ is metrically regular on $C \times (0, \ldots, 0) \subseteq X \times X^N$. See Ioffe's [33] for further information.

Theorem 3.7. If $C \neq \emptyset$ and $\{C_1, \ldots, C_N\}$ is metrically regular, then $\{C_1, \ldots, C_N\}$ is boundedly linearly regular.

Proof. We give details for $N = 2$ and the general case; the former is somewhat simpler.

Special case $N = 2$: The inequality of Definition 3.5 holds for all (b_1, b_2) sufficiently close to $(0, 0)$; in particular, the left-hand side is finite for such (b_1, b_2). Hence there exists $\delta > 0$ such that

$$(C_1 - b_1) \cap (C_2 - b_2) \neq \emptyset, \quad \text{whenever } \max\{\|b_1\|, \|b_2\|\} \leq \delta.$$

Now fix $b \in X$ with $\|b\| \leq \delta$ and set $b_1 := b$ and $b_2 := 0$. By the above, there exist $c_1 \in C_1$ and $c_2 \in C_2$ such that $c_1 - b_1 = c_2 - b_2$, or $b = c_1 - c_2 \in C_1 - C_2$. Denote the unit ball $\{x \in X : \|x\| \leq 1\}$ by B_X. Since b has been chosen arbitrarily in δB_X, it follows that $\delta B_X \subseteq C_1 - C_2$. Thus $0 \in \operatorname{int}(C_1 - C_2)$ and therefore $\{C_1, C_2\}$ is boundedly linearly regular by Fact 3.1.(ii).

General case $N \geq 2$: We work in $\mathbf{X} := X^N$ with $\mathbf{C} := C_1 \times \cdots \times C_N$ and $\Delta := \{\mathbf{x} = (x_i) \in \boldsymbol{\mathcal{X}} : x_1 = \cdots = x_N\}$.

Claim: $\exists \rho > 0$ such that $\rho B_{\mathbf{X}} \subseteq \mathbf{C} - \Delta$.

(This follows from a general result on metric regularity; see [33, Proposition 5.2]. We repeat the argument here for the reader's convenience.)

Fix an arbitrary $c \in C$. The set-valued map Ω from Remark 3.6 is metrically regular at $(c, \mathbf{0})$. Hence there is $K > 0$ and $\delta > 0$ such that

$$d\big(c, \Omega^{-1}(\mathbf{b})\big) \leq K d(\mathbf{b}, \Omega(c)), \quad \text{for all } \|\mathbf{b}\| \leq \delta.$$

Let $0 < \rho < \min\{\delta, \delta/K\}$ and fix an arbitrary $b \in \rho B_{\mathbf{X}}$. Since $\|\mathbf{b}\| \leq \delta$, we have

$$d\big(c, \Omega^{-1}(\mathbf{b})\big) \leq K d(\mathbf{b}, \Omega(c)) \leq K\|\mathbf{b} - \mathbf{0}\| \leq K\rho < \delta.$$

Hence there exists $x \in \Omega^{-1}(\mathbf{b})$ such that $\|c - x\| < \delta$. In particular, $\mathbf{b} \in \Omega(x) \subseteq \operatorname{ran} \Omega$. The Claim is thus proven.

The Claim and Fact 3.1.(ii) yield bounded linear regularity of $\{\mathbf{C}, \Delta\}$ in $\mathbf{X}$. We now argue as in the proof of [10, Theorem 5.19.(i)⇒(ii)] to conclude that $\{C_1, \ldots, C_N\}$ is boundedly linearly regular. ☐

Remark 3.8. A finite collection of zero subspaces in $X := \mathbb{R}$ shows that the converse of Theorem 3.7 is false in general. We thus observe that metric regularity is a genuinely more restrictive property than bounded linear regularity. In passing, we mention that Deutsch et al.'s property *strong CHIP* is genuinely less restrictive than bounded linear regularity. See [13] for details and further information.

Bounded linear regularity and the conical Open Mapping Theorem

In [12], bounded linear regularity was put in the broader context of convex optimization. Perhaps surprisingly, bounded linear regularity also relates to the classical Open Mapping Theorem. Since the latter is of general appeal, we now describe the connection.

Definition 3.9. [7] Suppose K is a closed convex cone in X and $T : X \to Y$ is a bounded linear operator to another real Hilbert space Y. Then T is *open relative to* K, if there exists $\delta > 0$ such that $\delta B_Y \cap T(K) \subseteq T(B_X \cap K)$.

Fact 3.10. (Conical Open Mapping Theorem) [7] Suppose K is a closed convex cone in X and $T : X \to Y$ is a bounded linear operator to another real Hilbert space Y. In $X \times Y$, set

$$K_1 := X \times 0, K_2 := \operatorname{gra}(T|_K),$$

i.e., K_2 is the graph of the restriction of T to K, and then take negative polar cones:

$$C_1 = K_1^{\ominus}, C_2 = K_2^{\ominus}.$$

Then T is open relative to $K \Leftrightarrow \{C_1, C_2\}$ is boundedly linearly regular.

Remark 3.11. Suppose $T : X \to Y$ is also onto. We obtain the classical Open Mapping Theorem through the following equivalences: T is an open mapping $\Leftrightarrow$ T is open relative to X $\Leftrightarrow$ $\big\{(X \times 0)^{\ominus}, (\operatorname{gra} T)^{\ominus}\big\}$ is boundedly linearly regular (Fact 3.10) $\Leftrightarrow$ $\big\{(X \times 0)^{\perp}, (\operatorname{gra} T)^{\perp}\big\}$ is boundedly linearly regular $\Leftrightarrow$ $(X \times 0) + \operatorname{gra} T$ is closed (Fact 3.2.(iv)) $\Leftrightarrow$ $X \times Y$ is closed (since T is onto), which is clearly true!

4. THE INCONSISTENT CASE

What happens if we apply the method of alternating projections to two (possibly non-intersecting) sets? This question is of great importance in applications (for instance, in Medical Imaging), where measurement errors may render the corresponding convex feasibility problem inconsistent.

The answer to this question is — up to the weak vs norm convergence problem (see Open Problem 1) — known:

Fact 4.1. (Dichotomy) [9] Suppose $N := 2$ and let $v := P_{\mathrm{cl}(C_2 - C_1)}(0)$. Then $x_{2n} - x_{2n-1} \to v$ and $x_{2n} - x_{2n+1} \to v$. Either $v \notin C_2 - C_1$ and $\|x_n\| \to +\infty$, or $v \in C_2 - C_1$ and

$$(x_{2n+1}) \text{ converges weakly to } e_1, \quad \text{and} \quad (x_{2n}) \text{ converges weakly to } e_2 = e_1 + v,$$

for some $e_1 \in E_1 = \{c_1 \in C_1 : d(c_1, C_2) = \|v\|\}$, and $e_2 \in E_2 = \{c_2 \in C_2 : d(c_2, C_1) = \|v\|\}$, where E_1, E_2 are closed convex sets with $E_1 + v = E_2$.

It would be highly satisfying to settle the following:

Open Problem 3. What is the behavior of the sequence of cyclic projections for three or more sets with empty intersection?

Some positive results are available, but the geometry of the problem is far from being understood. The crux of the problem is this: the vector v in Fact 4.1 is well-defined and important for the formulation of the behavior of the sequence of alternating projections even when the distance $d(C_1, C_2) = \inf\{\|c_1 - c_2\| : c_1 \in C_1, c_2 \in C_2\}$ is not attained, i.e., $v \notin C_2 - C_1$; however, it is not known how to define the vector(s) analogous to v in the N-set case. See [11] for an in-depth survey on the state of this problem.

5. RANDOM PROJECTIONS

In this section, we assume that

$$\boxed{C = \bigcap_{i=1}^{N} C_i \neq \emptyset} \text{ and } \boxed{r : \{1, 2, \dots\} \to \{1, 2, \dots, N\} \quad \text{is a } \textit{random map},}$$

i.e., it is onto and assumes every value infinitely often. We are interested in the *method of random projections*, i.e., the behavior of the sequence (z_n) defined by

$$\boxed{z_0 \in X, \qquad z_{n+1} := P_{r(n+1)} z_n, \quad \text{for } n \geq 0.}$$

If we let r be the "mod N" function (with remainders in $\{1, \dots, N\}$), then the sequence (z_n) is precisely a sequence of cyclic projections. In general, we just "roll a die" — this "N-die" could be unfair, but not to an extent where one set would be ignored eventually.

In 1965, Amemiya and Ando proved the following fundamental result.

Fact 5.1 (Amemiya and Ando). [1] Suppose each C_i is a subspace. Then the sequence of random projections converges *weakly* to a point in C.

For extensions to more general Banach spaces and mappings; see Dye, Khamsi, and Reich's [27] and Dye and Reich's [28].

In contrast to Fact 2.1, only weak convergence is asserted in Fact 5.1 — thus we are forced to ask the obvious question:

Open Problem 4. In Fact 5.1, can the convergence be only weak?

The situation is even less clear for general closed convex sets:

Open Problem 5. Does the sequence of random projections converge weakly to some point in C?

Remark 5.2. We point out that the case $N = 2$, which shaped our intuition earlier, is not helpful at all for these problems: indeed, since projections are idempotents, a sequence of random projections is essentially a sequence of alternating projections. Thus the answer to Open Problem 4 is a resounding "No" because of von Neumann's Fact 1.1, whereas Bregman's Fact 2.3 yields an affirmative answer to Open Problem 5.

In 1992, Dye and Reich showed that — for *three* sets — Open Problem 5 does have an affirmative answer:

Fact 5.3 (Dye and Reich). [28] If $N = 3$, then the sequence of random projections converges weakly to some point in C.

The reader is referred to Baillon and Bruck's [3] for further references on the random projection problem as well as the following stunning

Conjecture (Baillon and Bruck). [3] Suppose each C_i is a subspace. No matter what the random map r and the starting point z_0 is, there is a constant $K > 0$, depending only on N, such that

$$\|z_n - z_{n+l}\|^2 \leq K\big(\|z_n\|^2 - \|z_{n+l}\|^2\big), \quad \forall n \geq 0, \forall l \geq 0.$$

In fact, $K \leq \binom{N}{N/2}$.

If this conjecture is true, then every sequence of random projections is Cauchy in the subspace case; consequently, Open Problem 4 would be resolved.

Here is a positive result for the general case:

Fact 5.4. [5] Suppose $\{C_i : i \in I\}$ is boundedly regular, for all $\emptyset \neq I \subseteq \{1, \ldots, N\}$. Then each sequence of random projections converges *in norm* to a point in C.

The combination of Fact 5.4 with Fact 3.2.(iii) leads to a quite flexible norm convergence result in the subspace case.

6. ACCELERATION

In this last section, we assume that

$$\boxed{\text{each } C_i \text{ is a subspace, and } T = P_N P_{N-1} \cdots P_1.}$$

We now turn to an acceleration scheme that was first explicitly suggested by Gearhart and Koshy [29] in 1989. (It is, however, already implicit in the classical paper by Gubin, Polyak, and Raik [30], and closely related to work by Dax [24].)

Define

$$\boxed{A : X \to X : x \mapsto (1 - t_x)x + t_x Tx,}$$

where $t_x \in \mathbb{R}$ is chosen so that $\|A(x)\|$ is minimal. (There is a simple closed form for t_x.)

Then consider the sequence with

$$\boxed{\text{starting point } z_0 \in X, \text{ and } z^n := A^{n-1}(Tz_0), \text{ for all } n \geq 1.}$$

Fact 6.1. [15,14]

(i) (z_n) *always* converges *weakly* to $P_C z_0$.

(ii) (z_n) converges *in norm*, if $N = 2$.

(iii) (z_n) converges *linearly*, if $\{C_1, \ldots, C_N\}$ is boundedly linearly regular.

Once again, bounded linear regularity played a crucial role! We conclude with one last question:

Open Problem 6. Can (z_n) fail to converge in norm when $N \geq 3$ or when $\{C_1, \ldots, C_N\}$ is not boundedly linearly regular?

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REFERENCES

1. I. Amemiya and T. Andô. Convergence of random products of contractions in Hilbert space. *Acta Sci. Math. (Szeged)*, 26:239–244, 1965.
2. W. N. Anderson, Jr. and R. J. Duffin. Series and parallel addition of matrices. *J. Math. Anal. Appl.*, 26:576–594, 1969.
3. J.-B. Baillon and R. E. Bruck. On the random product of orthogonal projections in Hilbert space. 1998. Available at `http://mathlab.usc.edu/~bruck/research/papers/dvi/naca98.dvi`.
4. J. B. Baillon, R. E. Bruck, and S. Reich. On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces. *Houston J. Math.*, 4(1):1–9, 1978.
5. H. H. Bauschke. A norm convergence result on random products of relaxed projections in Hilbert space. *Trans. Amer. Math. Soc.*, 347(4):1365–1373, 1995.
6. H. H. Bauschke. *Projection algorithms and monotone operators.* PhD thesis, 1996. Available at `http://www.cecm.sfu.ca/preprints/1996pp.html`.
7. H. H. Bauschke and J. M. Borwein. Conical open mapping theorems and regularity. Proceedings of the Centre for Mathematics and its Applications (Australian National University). National Symposium on Functional Analysis, Optimization and Applications, March 1998.
8. H. H. Bauschke and J. M. Borwein. On the convergence of von Neumann's alternating projection algorithm for two sets. *Set-Valued Anal.*, 1(2):185–212, 1993.
9. H. H. Bauschke and J. M. Borwein. Dykstra's alternating projection algorithm for two sets. *J. Approx. Theory*, 79(3):418–443, 1994.
10. H. H. Bauschke and J. M. Borwein. On projection algorithms for solving convex feasibility problems. *SIAM Rev.*, 38(3):367–426, 1996.
11. H. H. Bauschke, J. M. Borwein, and A. S. Lewis. The method of cyclic projections for closed convex sets in Hilbert space. In *Recent developments in optimization theory and nonlinear analysis (Jerusalem, 1995)*, pages 1–38. Amer. Math. Soc., Providence, RI, 1997.
12. H. H. Bauschke, J. M. Borwein, and W. Li. Strong conical hull intersection property, bounded linear regularity, Jameson's property (G), and error bounds in convex optimization. *Math. Program. (Ser. A)*, 86(1):135–160, 1999.
13. H. H. Bauschke, J. M. Borwein, and P. Tseng. Bounded linear regularity, strong CHIP, and CHIP are distinct properties. To appear in *J. Convex Anal.*
14. H. H. Bauschke, F. Deutsch, H. Hundal, and S.-H. Park. Accelerating the convergence of the method of alternating projections. 1999. Submitted. Available at `http://www.cecm.sfu.ca/preprints/1999pp.html`.
15. H. H. Bauschke, F. Deutsch, H. Hundal, and S.-H. Park. Fejér monotonicity and weak convergence of an accelerated method of projections. In *Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999)*, pages 1–6. Canadian Math. Soc. Conference Proceedings Volume 27, 2000.
16. J. P. Boyle and R. L. Dykstra. A method for finding projections onto the intersection of convex sets in Hilbert spaces. In *Advances in order restricted statistical inference (Iowa City, Iowa, 1985)*, pages 28–47. Springer, Berlin, 1986.
17. L. M. Bregman. The method of successive projection for finding a common point of

convex sets. *Soviet Math. Dokl.*, 6:688–692, 1965.

18. R. E. Bruck and S. Reich. Nonexpansive projections and resolvents of accretive operators in Banach spaces. *Houston J. Math.*, 3(4):459–470, 1977.
19. Y. Censor and S. A. Zenios. *Parallel optimization.* Oxford University Press, New York, 1997.
20. P. L. Combettes. *The Convex Feasibility Problem in Image Recovery*, volume 95 of *Advances in Imaging and Electron Physics*, pages 155–270. Academic Press, 1996.
21. P. L. Combettes. Hilbertian convex feasibility problem: convergence of projection methods. *Appl. Math. Optim.*, 35(3):311–330, 1997.
22. P. L. Combettes. Fejér-monotonicity in convex optimization. In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of Optimization.* Kluwer, 2000.
23. P. L. Combettes. Quasi-Fejérian analysis of some optimization algorithms. In D. Butnariu, Y. Censor, and S. Reich, editors, *Inherently Parallel Algorithms in Feasibility and Optimization and their Applications (Haifa, 2000).* To appear.
24. A. Dax. Line search acceleration of iterative methods. *Linear Algebra Appl.*, 130:43–63, 1990. Linear algebra in image reconstruction from projections.
25. F. Deutsch. *Best approximation in inner product spaces.* Monograph. To appear.
26. F. Deutsch. The method of alternating orthogonal projections. In *Approximation theory, spline functions and applications (Maratea, 1991)*, pages 105–121. Kluwer Acad. Publ., Dordrecht, 1992.
27. J. Dye, M. A. Khamsi, and S. Reich. Random products of contractions in Banach spaces. *Trans. Amer. Math. Soc.*, 325(1):87–99, 1991.
28. J. M. Dye and S. Reich. Unrestricted iterations of nonexpansive mappings in Hilbert space. *Nonlinear Anal.*, 18(2):199–207, 1992.
29. W. B. Gearhart and M. Koshy. Acceleration schemes for the method of alternating projections. *J. Comput. Appl. Math.*, 26(3):235–249, 1989.
30. L. G. Gubin, B. T. Polyak, and E. V. Raik. The method of projections for finding the common point of convex sets. *Comput. Math. Math. Phys.*, 7(6):1–24, 1967.
31. I. Halperin. The product of projection operators. *Acta Sci. Math. (Szeged)*, 23:96–99, 1962.
32. A. J. Hoffman. On approximate solutions of systems of linear inequalities. *J. Research Nat. Bur. Standards*, 49:263–265, 1952.
33. A. D. Ioffe. Codirectional compactness, metric regularity and subdifferential calculus. In *Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999)*, pages 123–163. Canadian Math. Soc. Conference Proceedings Volume 27, 2000.
34. K. C. Kiwiel. Block-iterative surrogate projection methods for convex feasibility problems. *Linear Algebra Appl.*, 215:225–259, 1995.
35. K. C. Kiwiel and B. Łopuch. Surrogate projection methods for finding fixed points of firmly nonexpansive mappings. *SIAM J. Optim.*, 7(4):1084–1102, 1997.
36. Y. I. Merzlyakov. On a relaxation method of solving systems of linear inequalities. *Comput. Math. Math. Phys.*, 2(3):504–510, 1963.
37. S. Reich. A limit theorem for projections. *Linear and Multilinear Algebra*, 13(3):281–290, 1983.
38. J. von Neumann. *Functional Operators. II. The Geometry of Orthogonal Spaces.* Princeton University Press, Princeton, N. J., 1950. Annals of Math. Studies, no. 22.