Chapter 1
Chebyshev Sets, Klee Sets, and Chebyshev Centers with respect to Bregman Distances: Recent Results and Open Problems

Heinz H. Bauschke, Mason S. Macklem, and Xianfu Wang

Abstract In Euclidean spaces, the geometric notions of nearest-points map, farthest-points map, Chebyshev set, Klee set, and Chebyshev center are well known and well understood. Since early works going back to the 1930s, tremendous theoretical progress has been made, mostly by extending classical results from Euclidean space to Banach space settings. In all these results, the distance between points is induced by some underlying norm. Recently, these notions have been revisited from a different viewpoint in which the discrepancy between points is measured by Bregman distances induced by Legendre functions. The associated framework covers the well known Kullback-Leibler divergence and the Itakura-Saito distance. In this survey, we review known results and we present new results on Klee sets and Chebyshev centers with respect to Bregman distances. Examples are provided and connections to recent work on Chebyshev functions are made. We also identify several intriguing open problems.

Key words: Bregman distance, Chebyshev center, Chebyshev function, Chebyshev point of a function, Chebyshev set, convex function, farthest point, Fenchel conjugate, Itakura-Saito distance, Klee set, Klee function, Kullback-Leibler divergence, Legendre function, nearest point, projection.

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Heinz H. Bauschke
Mathematics, University of British Columbia Okanagan, Kelowna, B.C. V1V 1V7, CANADA
e-mail: heinz.bauschke@ubc.ca

Mason S. Macklem
Mathematics, University of British Columbia Okanagan, Kelowna, B.C. V1V 1V7, CANADA
e-mail: mason.macklem@ubc.ca

Xianfu Wang
Mathematics, University of British Columbia Okanagan, Kelowna, B.C. V1V 1V7, CANADA
e-mail: shawn.wang@ubc.ca

1.1 Introduction

Legendre Functions and Bregman Distances

Throughout, we assume that

$$X = \mathbb{R}^n \text{ is the standard Euclidean space with inner product } \langle \cdot, \cdot \rangle, \quad (1.1)$$

with induced norm $\|\cdot\| : x \mapsto \sqrt{\langle x, x \rangle}$, and with metric $(x, y) \mapsto \|x - y\|$. In addition, it is assumed that

$$f : X \to \,]-\infty, +\infty] \text{ is a convex function of Legendre type,} \quad (1.2)$$

also referred as a Legendre function. We assume the reader is familiar with basic results and standard notation from Convex Analysis; see, e.g., [33, 34, 40]. In particular, f^* denotes the Fenchel conjugate of f, and $\operatorname{int} \operatorname{dom} f$ is the interior of the domain of f. For a subset C of X, $\overline{C}$ stands for the closure of C, $\operatorname{conv} C$ for the convex hull of C, and ι_C for the indicator function of C, i.e., $\iota_C(x) = 0$, if $x \in C$ and $\iota_C(x) = +\infty$, if $x \in X \smallsetminus C$. Now set

$$U = \operatorname{int} \operatorname{dom} f. \quad (1.3)$$

Example 1.1.1 (Legendre functions) The following are Legendre functions [1], each evaluated at a point $x \in X$.

(i) *halved energy:* $f(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2}\sum_j x_j^2$.

(ii) *negative entropy:* $f(x) = \begin{cases} \sum_j \left(x_j \ln(x_j) - x_j \right), & \text{if } x \geq 0; \\ +\infty, & \text{otherwise.} \end{cases}$

(iii) *negative logarithm:* $f(x) = \begin{cases} -\sum_j \ln(x_j), & \text{if } x > 0; \\ +\infty, & \text{otherwise.} \end{cases}$

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[1] Here and elsewhere, inequalities between vectors in $\mathbb{R}^n$ are interpreted coordinate-wise.

Note that $U = \mathbb{R}^n$ in (i), whereas $U = \mathbb{R}^n_{++}$ in (ii) and (iii).

Further examples of Legendre functions can be found in, e.g., [2,4,12,33].

Fact 1.1.2 (Rockafellar) (See [33, Theorem 26.5].) *The gradient map ∇f is a continuous bijection between* $\operatorname{int}\operatorname{dom} f$ *and* $\operatorname{int}\operatorname{dom} f^*$, *with continuous inverse map* $(\nabla f)^{-1} = \nabla f^*$. *Furthermore, f^* is also a convex function of Legendre type.*

Given $x \in U$ and $C \subseteq U$, it will be convenient to write

$$
\begin{aligned}
x^* &= \nabla f(x), & (1.4)\\
C^* &= \nabla f(C), & (1.5)\\
U^* &= \operatorname{int}\operatorname{dom} f^*, & (1.6)
\end{aligned}
$$

and similarly for other vectors and sets in U. Note that we used Fact 1.1.2 for (1.6).

While the Bregman distance defined next is not a distance in the sense of metric topology, it does possess some good properties that allow it to measure the discrepancy between points in U.

Definition 1.1.3 (Bregman distance) (See [13,15,16].) *The* Bregman distance *with respect to f, written D_f or simply D, is the function*

$$
D\colon X \times X \to [0,+\infty]\colon (x,y) \mapsto
\begin{cases}
f(x) - f(y) - \langle \nabla f(y), x - y \rangle, & \text{if } y \in U;\\
+\infty, & \text{otherwise.}
\end{cases}
\quad (1.7)
$$

Fact 1.1.4 (See [2, Proposition 3.2.(i) and Theorem 3.7.(iv)&(v)].) *Let x and y be in U. Then the following hold.*

(i) $D_f(x,y) = f(x) + f^*(y^*) - \langle y^*, x \rangle = D_{f^*}(y^*, x^*)$.

(ii) $D_f(x,y) = 0 \Leftrightarrow x = y \Leftrightarrow x^* = y^* \Leftrightarrow D_{f^*}(x^*, y^*) = 0$.

Example 1.1.5 The Bregman distances corresponding to the Legendre functions of Example 1.1.1 between two points x and y in X are as follows.

(i) $D(x,y) = \frac{1}{2}\|x - y\|^2$.

(ii) $D(x,y) = \begin{cases} \sum_j \left(x_j \ln(x_j/y_j) - x_j + y_j \right), & \text{if } x \geq 0 \text{ and } y > 0;\\ +\infty, & \text{otherwise.} \end{cases}$

(iii) $D(x,y) = \begin{cases} \sum_j \left(\ln(y_j/x_j) + x_j/y_j - 1 \right), & \text{if } x > 0 \text{ and } y > 0;\\ +\infty, & \text{otherwise.} \end{cases}$

These Bregman distances are also known as (i) the *halved Euclidean distance squared*, (ii) the *Kullback-Leibler divergence*, and (iii) the *Itakura-Saito distance*, respectively.

From now on, we assume that C is a subset of X such that

$$C \text{ is closed} \quad \text{and} \quad \varnothing \neq C \subseteq U. \tag{1.8}$$

The *power set* (the set of all subsets) of C is denoted by 2^C.

We are now in a position to introduce the various geometric notions.

Nearest Distance, Nearest Points, and Chebyshev Sets

Definition 1.1.6 (Bregman nearest-distance function and nearest-points map)
The left Bregman nearest-distance function *with respect to C is*

$$\overleftarrow{D}_C : X \to [0, +\infty] : y \mapsto \inf_{x \in C} D(x, y), \tag{1.9}$$

and the left Bregman nearest-points map[2] *with respect to C is*

$$\overleftarrow{P}_C : X \to 2^C : y \mapsto \left\{ x \in C \mid D(x,y) = \overleftarrow{D}_C(y) < +\infty \right\}. \tag{1.10}$$

The right Bregman nearest-distance *and the* right Bregman nearest-point map *with respect to C are*

$$\overrightarrow{D}_C : X \to [0, +\infty] : x \mapsto \inf_{y \in C} D(x, y) \tag{1.11}$$

and

$$\overrightarrow{P}_C : X \to 2^C : x \mapsto \left\{ y \in C \mid D(x,y) = \overrightarrow{D}_C(x) < +\infty \right\}, \tag{1.12}$$

respectively. If we need to emphasize the underlying Legendre function f, then we write $\overleftarrow{D}_{f,C}$, $\overleftarrow{P}_{f,C}$, $\overrightarrow{D}_{f,C}$, and $\overrightarrow{P}_{f,C}$.

Definition 1.1.7 (Chebyshev sets) *The set C is a* left Chebyshev set *with respect to the Bregman distance, or simply $\overleftarrow{D}$-Chebyshev, if for every $y \in U$, $\overleftarrow{P}_C(y)$ is a singleton. Similarly, the set C is a* right Chebyshev set *with respect to the Bregman distance, or simply $\overrightarrow{D}$-Chebyshev, if for every $x \in U$, $\overrightarrow{P}_C(x)$ is a singleton.*

Remark 1.1.8 (Classical Bunt-Motzkin result) Assume that f is the halved energy as in Example 1.1.1(i). Since the halved Euclidean distance squared (see Example 1.1.5(i)) is symmetric, the left and right (Bregman) nearest distances coincide,

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[2] This operator, which has turned out to be quite useful in Optimization and which has found many applications (for a recent one, see [32]), is often referred to as the Bregman projection.

as do the corresponding nearest-point maps. Furthermore, the set C is Chebyshev if and only if for every $z \in X$, the metric[3] projection $P_C(z)$ is a singleton. It is well known that if C is convex, then C is Chebyshev. In the mid-1930s, Bunt [14] and Motzkin [28] showed independently that the following converse holds:

$$C \text{ is Chebyshev} \implies C \text{ is convex.} \tag{1.13}$$

For other works in this direction, see, e.g., [1, 9–11, 17, 22, 24, 25, 35–37]. It is still unknown whether or not (1.13) holds in general Hilbert spaces. We review corresponding results for the present Bregman setting in Section 1.3 below.

Farthest Distance, Farthest Points, and Klee Sets

Definition 1.1.9 (Bregman farthest-distance function and farthest-points map)

The left Bregman farthest-distance function *with respect to C is*

$$\overleftarrow{F}_C \colon X \to [0, +\infty] \colon y \mapsto \sup_{x \in C} D(x, y), \tag{1.14}$$

and the left Bregman farthest-points map *with respect to C is*

$$\overleftarrow{Q}_C \colon X \to 2^C \colon y \mapsto \left\{ x \in C \mid D(x, y) = \overleftarrow{F}_C(y) < +\infty \right\}. \tag{1.15}$$

Similarly, the right Bregman farthest-distance function *with respect to C is*

$$\overrightarrow{F}_C \colon X \to [0, +\infty] \colon x \mapsto \sup_{y \in C} D(x, y), \tag{1.16}$$

and the right Bregman farthest-points map *with respect to C is*

$$\overrightarrow{Q}_C \colon X \to 2^C \colon x \mapsto \left\{ y \in C \mid D(x, y) = \overrightarrow{F}_C(x) < +\infty \right\}. \tag{1.17}$$

If we need to emphasize the underlying Legendre function f, then we write $\overleftarrow{F}_{f,C}$, $\overleftarrow{Q}_{f,C}$, $\overrightarrow{F}_{f,C}$, *and* $\overrightarrow{Q}_{f,C}$.

Definition 1.1.10 (Klee sets) *The set C is a* left Klee set *with respect to the Bregman distance, or simply* $\overleftarrow{D}$-Klee, *if for every $y \in U$, $\overleftarrow{Q}_C(y)$ is a singleton. Similarly, the set C is a* right Klee set *with respect to the right Bregman distance, or simply* $\overrightarrow{D}$-Klee, *if for every $x \in U$, $\overrightarrow{Q}_C(x)$ is a singleton.*

Remark 1.1.11 (Classical Klee result) Assume again that f is the halved energy as in Example 1.1.1(i). Then the left and right (Bregman) farthest-distance func-

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[3] The metric projection is the nearest-points map with respect to the Euclidean distance.

tions coincide, as do the corresponding farthest-points maps. Furthermore, the set C is Klee if and only if for every $z \in X$, the metric farthest-points map $Q_C(z)$ is a singleton. It is obvious that if C is a singleton, then C is Klee. In 1961, Klee [27] showed the following converse:

$$C \text{ is Klee } \implies C \text{ is a singleton.} \tag{1.18}$$

See, e.g., also [1,11,17,23–25,29,39]. Once again, it is still unknown whether or not (1.18) remains true in general Hilbert spaces. The present Bregman-distance setting is reviewed in Section 1.4 below.

Chebyshev Radius and Chebyshev Center

Definition 1.1.12 (Chebyshev radius and Chebyshev center)
The left $\overleftarrow{D}$-Chebyshev radius of C is

$$\overleftarrow{r}_C = \inf_{y \in U} \overleftarrow{F}_C(y) \tag{1.19}$$

and the left $\overleftarrow{D}$-Chebyshev center of C is

$$\overleftarrow{Z}_C = \left\{ y \in U \mid \overleftarrow{F}_C(y) = \overleftarrow{r}_C < +\infty \right\}. \tag{1.20}$$

Similarly, the right $\overrightarrow{D}$-Chebyshev radius of C is

$$\overrightarrow{r}_C = \inf_{x \in U} \overrightarrow{F}_C(x) \tag{1.21}$$

and the right $\overrightarrow{D}$-Chebyshev center of C is

$$\overrightarrow{Z}_C = \left\{ x \in U \mid \overrightarrow{F}_C(x) = \overrightarrow{r}_C < +\infty \right\}. \tag{1.22}$$

If we need to emphasize the underlying Legendre function f, then we write $\overleftarrow{r}_{f,C}$, $\overleftarrow{Z}_{f,C}$, $\overrightarrow{r}_{f,C}$, and $\overrightarrow{Z}_{f,C}$.

Remark 1.1.13 (Classical Garkavi-Klee result) Again, assume that f is the halved energy as in Example 1.1.1(i) so that the left and right (Bregman) farthest-distance functions coincide, as do the corresponding farthest-points maps. Furthermore, assume that C is bounded. In the 1960s, Garkavi [19] and Klee [26] proved that the Chebyshev center is a singleton, say $\{z\}$, which is characterized by

$$z \in \operatorname{conv} Q_C(z). \tag{1.23}$$

See also [30,31] and Section 1.5 below. In passing, we note that Chebyshev centers are also utilized in Fixed Point Theory; see, e.g., [20, Chapter 4].

Goal of the Paper

The aim of this survey is three-fold. First, we review recent results concerning Chebyshev sets, Klee sets, and Chebyshev centers with respect to Bregman distances. Secondly, we provide some new results and examples on Klee sets and Chebyshev centers. Thirdly, we formulate various tantalizing open problems on these notions as well as on the related concepts of Chebyshev functions.

Organization of the Paper

The remainder of the paper is organized as follows. In Section 1.2, we record auxiliary results which will make the derivation of the main results more structured. Chebyshev sets and corresponding open problems are discussed in Section 1.3. In Section 1.4, we review results and open problems for Klee sets, and we also present a new result (Theorem 1.4.3) concerning left Klee sets. Chebyshev centers are considered in Section 1.5, where we also provide a characterization of left Chebyshev centers (Theorem 1.5.2). Chebyshev centers are illustrated by two examples in Section 1.6. Recent related results on variations of Chebyshev sets and Klee sets are considered in Section 1.7. Along our journey, we pose several questions that we list collectively in the final Section 1.8.

1.2 Auxiliary Results

For the reader's convenience, we present the following two results which are implicitly contained in [7] and [8].

Lemma 1.2.1 *Let x and y be in C. Then the following hold.*

(i) $\overleftarrow{D}_{f,C}(y) = \overrightarrow{D}_{f^*,C^*}(y^*)$ *and* $\overrightarrow{D}_{f,C}(x) = \overleftarrow{D}_{f^*,C^*}(x^*)$.

(ii) $\overleftarrow{P}_{f,C}\big|_U = \nabla f^* \circ \overrightarrow{P}_{f^*,C^*} \circ \nabla f$ *and* $\overrightarrow{P}_{f,C}\big|_U = \nabla f^* \circ \overleftarrow{P}_{f^*,C^*} \circ \nabla f$.

(iii) $\overleftarrow{P}_{f^*,C^*}\big|_{U^*} = \nabla f \circ \overrightarrow{P}_{f,C} \circ \nabla f^*$ *and* $\overrightarrow{P}_{f^*,C^*}\big|_{U^*} = \nabla f \circ \overleftarrow{P}_{f,C} \circ \nabla f^*$.

Proof. This follows from Fact 1.1.2, Fact 1.1.4(i), and Definition 1.1.6. (See also [7, Proposition 7.1].) ∎

Lemma 1.2.2 *Let x and y be in C. Then the following hold.*

(i) $\overleftarrow{F}_{f,C}(y) = \overrightarrow{F}_{f^*,C^*}(y^*)$ *and* $\overrightarrow{F}_{f,C}(x) = \overleftarrow{F}_{f^*,C^*}(x^*)$.

(ii) $\overleftarrow{Q}_{f,C}\big|_U = \nabla f^* \circ \overrightarrow{Q}_{f^*,C^*} \circ \nabla f$ *and* $\overrightarrow{Q}_{f,C}\big|_U = \nabla f^* \circ \overleftarrow{Q}_{f^*,C^*} \circ \nabla f$.

(iii) $\overleftarrow{Q}_{f^*,C^*}\big|_{U^*} = \nabla f \circ \overrightarrow{Q}_{f,C} \circ \nabla f^*$ *and* $\overrightarrow{Q}_{f^*,C^*}\big|_{U^*} = \nabla f \circ \overleftarrow{Q}_{f,C} \circ \nabla f^*$.

Proof. This follows from Fact 1.1.2, Fact 1.1.4(i), and Definition 1.1.9. (See also [8, Proposition 7.1].) ∎

The next observation on the duality of Chebyshev radii and Chebyshev centers is new.

Lemma 1.2.3 *The following hold.*

(i) $\overleftarrow{r}_{f,C} = \overrightarrow{r}_{f^*,C^*}$ *and* $\overrightarrow{r}_{f,C} = \overleftarrow{r}_{f^*,C^*}$.

(ii) $\overleftarrow{Z}_{f,C} = \nabla f^*(\overrightarrow{Z}_{f^*,C^*})$ *and* $\overrightarrow{Z}_{f,C} = \nabla f^*(\overleftarrow{Z}_{f^*,C^*})$.

(iii) $\overleftarrow{Z}_{f^*,C^*} = \nabla f(\overrightarrow{Z}_{f,C})$ *and* $\overrightarrow{Z}_{f^*,C^*} = \nabla f(\overleftarrow{Z}_{f,C})$.

(iv) $\overleftarrow{Z}_{f,C}$ *is a singleton* $\Leftrightarrow$ $\overrightarrow{Z}_{f^*,C^*}$ *is a singleton.*

(v) $\overrightarrow{Z}_{f,C}$ *is a singleton* $\Leftrightarrow$ $\overleftarrow{Z}_{f^*,C^*}$ *is a singleton.*

Proof. (i): Using Definition 1.1.12 and Lemma 1.2.2(i), we see that

$$\overleftarrow{r}_{f,C} = \inf_{y \in U} \overleftarrow{F}_C(y) = \inf_{y^* \in U^*} \overrightarrow{F}_{C^*}(y^*) = \overrightarrow{r}_{f^*,C^*} \tag{1.24}$$

and that

$$\overrightarrow{r}_{f,C} = \inf_{y \in U} \overrightarrow{F}_C(y) = \inf_{y^* \in U^*} \overleftarrow{F}_{C^*}(y^*) = \overleftarrow{r}_{f^*,C^*}. \tag{1.25}$$

(ii)&(iii): Let $z \in U$. Using (i) and Lemma 1.2.2(i), we see that

$$z \in \overleftarrow{Z}_{f,C} \Leftrightarrow \overleftarrow{F}_{f,C}(z) = \overleftarrow{r}_{f,C} \Leftrightarrow \overrightarrow{F}_{f^*,C^*}(z^*) = \overrightarrow{r}_{f^*,C^*} \Leftrightarrow z^* \in \overrightarrow{Z}_{f^*,C^*}. \tag{1.26}$$

This verifies $\overleftarrow{Z}_{f,C} = \nabla f^*(\overrightarrow{Z}_{f^*,C^*})$ and $\overrightarrow{Z}_{f^*,C^*} = \nabla f(\overleftarrow{Z}_{f,C})$. The remaining identities follow similarly.

(iv)&(v): Clear from (ii)&(iii) and Fact 1.1.2. ∎

The following two results play a key role for studying the single-valuedness of $\overrightarrow{P}_{f,C}$ via $\overleftarrow{P}_{f^*,C^*}$ and $\overrightarrow{Q}_{f,C}$ via $\overleftarrow{Q}_{f^*,C^*}$ by duality.

Lemma 1.2.4 *Let V and W be nonempty open subsets of X, and let $T : V \to W$ be a homeomorphism, i.e., T is a bijection and both T and T^{-1} are continuous. Furthermore, let G be a residual* [4] *subset of V. Then $T(G)$ is a residual subset of W.*

Proof. As G is residual, there exist sequence of dense open subsets $(O_k)_{k \in \mathbb{N}}$ of V such that $G \supseteq \bigcap_{k \in \mathbb{N}} O_k$. Then $T(G) \supseteq T(\bigcap_{k \in \mathbb{N}} O_k) = \bigcap_{k \in \mathbb{N}} T(O_k)$. Since $T : V \to W$ is a homeomorphism and each O_k is dense in V, we see that each $T(O_k)$ is open and dense in W. Therefore, $\bigcap_{k \in \mathbb{N}} T(O_k)$ is a dense G_δ subset in W. ∎

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[4] also known as "second category"

Lemma 1.2.5 *Let V be a nonempty open subset of X, and let $T : V \to \mathbb{R}^n$ be locally Lipschitz. Furthermore, let S be a subset of V that has Lebesgue measure zero. Then $T(S)$ has Lebesgue measure zero as well.*

Proof. Denote the closed unit ball in X by $\mathbb{B}$. For every $y \in V$, let $r(y) > 0$ be such that T is Lipschitz continuous with constant $c(y)$ on the open ball $O(y)$ centered at y of radius $r(y)$. In this proof we denote the Lebesgue measure by λ. Let K be a compact subset of X. To show that $T(S)$ has Lebesgue measure zero, it suffices to show that $\lambda(T(K \cap S)) = 0$ because

$$\lambda\big(T(S)\big) = \lambda\left(T\left(\bigcup_{k \in \mathbb{N}} S \cap k\mathbb{B}\right)\right) \le \sum_{k \in \mathbb{N}} \lambda\big(T(S \cap k\mathbb{B})\big). \tag{1.27}$$

The Heine-Borel theorem provides a finite subset $\{y_1, \dots, y_m\}$ of V such that

$$K \subseteq \bigcup_{j=1}^{m} O(y_j). \tag{1.28}$$

We now proceed using a technique implicit in the proof of [21, Corollary 1]. Set $c = \max\{c_1, c_2, \dots, c_m\}$. Given $\varepsilon > 0$, there exists an open subset G of X such that $G \supseteq K \cap S$ and $\lambda(G) < \varepsilon$. For each $y \in K \cap S$, let $Q(y)$ be an open cubic interval centered at y of semi-edge length $s(y) > 0$ such that

$$(\exists j \in \{1, \dots, m\}) \quad Q(y) \subseteq G \cap O(y_j). \tag{1.29}$$

Then for each $x \in Q(y)$, we have

$$\|Tx - Ty\| \le c\|x - y\| \le c\sqrt{n}s(y). \tag{1.30}$$

Hence the image of $Q(y)$ by T, $T(Q(y))$, is contained in a cubic interval — which we denote by $Q^*(Ty)$ — of center Ty and with semi-edge length $c\sqrt{n}s(y)$. Applying the Besicovitch Covering Theorem, we see that there exists a sequence $(Q_k)_{k \in \mathbb{N}}$ chosen among the open covering $(Q(y))_{y \in K \cap S}$ such that

$$K \cap S \subseteq \bigcup_{k \in \mathbb{N}} Q_k \quad \text{and} \quad \sum_{k \in \mathbb{N}} \chi_{Q_k} \le \theta, \tag{1.31}$$

where χ_{Q_k} stands for the characteristic function of Q_k and where the constant θ only depends on the dimension of X. Thus,

$$T(K \cap S) \subseteq T\left(\bigcup_{k \in \mathbb{N}} Q_k\right) = \bigcup_{k \in \mathbb{N}} T(Q_k) \subseteq \bigcup_{k \in \mathbb{N}} Q_k^*. \tag{1.32}$$

Now set $d = (c\sqrt{n})^n$ so that $\lambda(Q_k^*) \le d\lambda(Q_k)$. Then, using (1.29) and (1.31), we see that

$$\lambda\left(\cup_{k\in\mathbb{N}}Q_k^*\right) \le \sum_{k\in\mathbb{N}}\lambda(Q_k^*) \le d\sum_{k\in\mathbb{N}}\lambda(Q_k) = d\sum_{k\in\mathbb{N}}\int\chi_{Q_k} = d\int\sum_{k\in\mathbb{N}}\chi_{Q_k} \le d\theta\lambda(G)$$

$$\le d\theta\varepsilon. \tag{1.33}$$

Since ε was chosen arbitrarily, we conclude that $\lambda(T(K\cap S)) = 0$.

Alternatively, one may argue as follows starting from (1.28). We have $K\cap S \subseteq \left(\bigcup_{j=1}^m O(y_j)\right)\cap S = \bigcup_{j=1}^m O(y_j)\cap S$ so that

$$T(K\cap S) \subseteq \bigcup_{j=1}^m T(O(y_j)\cap S). \tag{1.34}$$

Since T is Lipschitz on each $O(y_j)$ with constant $c(y_j)$ and since $\lambda(O(y_j)\cap S) = 0$, we apply [18, Proposition 262D, page 286] and conclude that $\lambda(T(O(y_j)\cap S)) = 0$. Therefore, $\lambda(T(K\cap S)) = 0$ by (1.34). ∎

1.3 Chebyshev Sets

We start by reviewing the strongest known results concerning left and right Chebyshev sets with respect to Bregman distances.

Fact 1.3.1 ($\overleftarrow{D}$-Chebyshev sets) (See [7, Theorem 4.7].) *Suppose that f is supercoercive[5] and that C is $\overleftarrow{D}$-Chebyshev. Then C is convex.*

Fact 1.3.2 ($\overrightarrow{D}$-Chebyshev sets) (See [7, Theorem 7.3].) *Suppose that* dom $f = X$, *that $\overline{C^*} \subseteq U^*$, and that C is $\overrightarrow{D}$-Chebyshev. Then C^* is convex.*

It is not known whether or not Fact 1.3.1 and Fact 1.3.2 are the best possible results. For instance, is the assumption on supercoercivity in Fact 1.3.1 really necessarily? Similarly, do we really require full domain of f in Fact 1.3.2?

Example 1.3.3 (See [7, Example 7.5].) Suppose that $X = \mathbb{R}^2$, that f is the negative entropy (see Example 1.1.1(ii)), and that

$$C = \left\{(e^\lambda, e^{2\lambda}) \mid \lambda \in [0,1]\right\}. \tag{1.35}$$

Then f is supercoercive and C is a *nonconvex $\overrightarrow{D}$*-Chebyshev set.

Example 1.3.3 is somewhat curious — not only does it illustrate that the right-Chebyshev-set counterpart of Fact 1.3.1 fails but it also shows that the conclusion of Fact 1.3.2 may hold even though f is not assumed to have full domain.

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[5] By [2, Proposition 2.16] and [33, Corollary 14.2.2], f is supercoercive $\Leftrightarrow \lim_{\|x\|\to+\infty}\frac{f(x)}{\|x\|} = +\infty \Leftrightarrow$ dom $f^* = X \Rightarrow 0 \in$ int dom $f^* \Leftrightarrow \lim_{\|x\|\to+\infty} f(x) = +\infty \Leftrightarrow f$ is coercive.

Fact 1.3.4 (See [6, Lemma 3.5].) *Suppose that f is the negative entropy (see Example 1.1.1(ii)) and that C is convex. Then C is $\overrightarrow{D}$-Chebyshev.*

Fact 1.3.4 raises two intriguing questions. Apart from the case of quadratic functions, are there instances of f where f has full domain and where every closed convex subset of U is $\overrightarrow{D}$-Chebyshev? Because of Fact 1.3.2, an affirmative answer to this question would imply that ∇f is a (quite surprising) *nonaffine yet convexity-preserving* transformation. Combining Example 1.3.3 and Fact 1.3.4, we deduce that — when working with the negative entropy — if C is convex, then C is $\overrightarrow{D}$-Chebyshev but *not* vice versa. Is it possible to describe the $\overrightarrow{D}$-Chebyshev sets in this setting?

We also note that C is "nearly $\overleftarrow{D}$-Chebyshev" in the following sense.

Fact 1.3.5 (See [7, Corollary 5.6].) *Suppose that f is supercoercive, that f is twice continuously differentiable, and that for every $y \in U$, $\nabla^2 f(y)$ is positive definite. Then $\overleftarrow{P}_C$ is almost everywhere and generically[6] single-valued on U.*

It would be interesting to see whether or not supercoercivity is essential in Fact 1.3.5. By duality, we obtain the following result on the single-valuedness of $\overrightarrow{P}_{f,C}$.

Corollary 1.3.6 *Suppose that f has full domain, that f^* is twice continuously differentiable, and that $\nabla^2 f^*(y)$ is positive definite for every $y \in U^*$. Then $\overrightarrow{P}_{f,C}$ is almost everywhere and generically single-valued on U.*

Proof. By Lemma 1.2.1(ii), $\overrightarrow{P}_{f,C}\big|_U = \nabla f^* \circ \overleftarrow{P}_{f^*,C^*} \circ \nabla f$. Fact 1.3.5 states that $\overleftarrow{P}_{f^*,C^*}$ is almost everywhere and generically single-valued on U^*. Since f^* is twice continuously differentiable, it follows from the Mean Value Theorem that ∇f^* is locally Lipschitz. Since $(\nabla f)^{-1} = \nabla f^*$ is a locally Lipschitz homeomorphism from U^* to U, the conclusion follows from Lemma 1.2.4 and Lemma 1.2.5. ∎

1.4 Klee Sets

Previously known were the following two results.

Fact 1.4.1 ($\overleftarrow{D}$-**Klee sets**) (See [8, Theorem 4.4].) *Suppose that f is supercoercive, that C is bounded, and that C is $\overleftarrow{D}$-Klee. Then C is a singleton.*

Fact 1.4.2 ($\overrightarrow{D}$-**Klee sets**) (See [5, Theorem 3.2].) *Suppose that C is bounded and that C is $\overrightarrow{D}$-Klee. Then C is a singleton.*

---

[6] That is, the set S of points $y \in U$ where $\overleftarrow{P}_C(y)$ is *not* a singleton is very small both in measure theory (S has measure 0) and in category theory (S is meager/first category).

Fact 1.4.1 immediately raises the question whether or not supercoercivity is really an essential hypothesis. Fortunately, thanks to Fact 1.4.2, which was recently proved for general Legendre functions without any further assumptions, we are now able to present a new result which removes the supercoercivity assumption in Fact 1.4.1.

Theorem 1.4.3 ($\overleftarrow{D}$-Klee sets revisited) *Suppose that C is bounded and that C is $\overleftarrow{D}$-Klee. Then C is a singleton.*

Proof. On the one hand, since C is compact, Fact 1.1.2 implies that C^* is compact. On the other hand, by Lemma 1.2.2(iii), the set C^* is $\overrightarrow{D}_{f^*}$-Klee. Altogether, we deduce from Fact 1.4.2 (applied to f^* and C^*) that C^* is a singleton. Therefore, C is a singleton by Fact 1.1.2. ∎

Similarly to the setting of Chebyshev sets, the set C is "nearly $\overleftarrow{D}$-Klee" in the following sense.

Fact 1.4.4 (See [7, Corollary 5.2.(ii)].) *Suppose that f is supercoercive, that f is twice continuously differentiable, that for every $y \in U$, $\nabla^2 f(y)$ is positive definite, and that C is bounded. Then $\overleftarrow{Q}_C$ is almost everywhere and generically single-valued on U.*

Again, it would be interesting to see whether or not supercoercivity is essential in Fact 1.4.4. Similarly to the proof of Corollary 1.3.6, we obtain the following result on the single-valuedness of $\overrightarrow{Q}_{f,C}$.

Corollary 1.4.5 *Suppose that f has full domain, that f^* is twice continuously differentiable, that $\nabla^2 f^*(y)$ is positive definite for every $y \in U^*$, and that C is bounded. Then $\overrightarrow{Q}_{f,C}$ is almost everywhere and generically single-valued on U.*

1.5 Chebyshev Centers: Uniqueness and Characterization

Fact 1.5.1 ($\overrightarrow{D}$-Chebyshev centers) (See [5, Theorem 4.4].) *Suppose that C is bounded. Then the right Chebyshev center with respect to C is a singleton, say $\overrightarrow{Z}_C = \{x\}$, and x is characterized by*

$$x \in \nabla f^* \big(\operatorname{conv} \nabla f(\overrightarrow{Q}_C(x)) \big). \tag{1.36}$$

We now present a corresponding new result on the left Chebyshev center.

Theorem 1.5.2 ($\overleftarrow{D}$-Chebyshev centers) *Suppose that C is bounded. Then the left Chebyshev center with respect to C is a singleton, say $\overleftarrow{Z}_C = \{y\}$, and y is characterized by*

$$y \in \operatorname{conv} \overleftarrow{Q}_C(y). \tag{1.37}$$

Proof. By Lemma 1.2.3(ii),

$$\overleftarrow{Z}_{f,C} = \nabla f^*\big(\overrightarrow{Z}_{f^*,C^*}\big). \tag{1.38}$$

Now C^* is a bounded subset of U^* because of the compactness of C and Fact 1.1.2. Applying Fact 1.5.1 to f^* and C^*, we obtain that $\overrightarrow{Z}_{f^*,C^*} = \{y^*\}$ for some $y^* \in U^*$ and that y^* is characterized by

$$y^* \in \nabla f\big(\mathrm{conv}\,\nabla f^*(\overrightarrow{Q}_{f^*,C^*}(y^*))\big). \tag{1.39}$$

By (1.38), $\overleftarrow{Z}_{f,C} = \nabla f^*\big(\overrightarrow{Z}_{f^*,C^*}\big) = \{\nabla f^*(y^*)\} = \{y\}$ is a singleton. Moreover, using Lemma 1.2.2(ii), we see that the characterization (1.39) becomes

$$\begin{aligned}
\overleftarrow{Z}_{f,C} = \{y\} &\Leftrightarrow y^* \in \nabla f\big(\mathrm{conv}\,\nabla f^*(\overrightarrow{Q}_{f^*,C^*}(y^*))\big) \\
&\Leftrightarrow \nabla f^*(y^*) \in \mathrm{conv}\,\nabla f^*(\overrightarrow{Q}_{f^*,C^*}(y^*)) \\
&\Leftrightarrow y \in \mathrm{conv}\,\nabla f^*(\overrightarrow{Q}_{f^*,C^*}(\nabla f(y))) \\
&\Leftrightarrow y \in \mathrm{conv}\,\overleftarrow{Q}_{f,C}(y), \tag{1.40}
\end{aligned}$$

as claimed.   ∎

Remark 1.5.3 The proof of Fact 1.5.1 does not carry over directly to the setting of Theorem 1.5.2. Indeed, one key element in that proof was to realize that the right farthest distance function

$$\overrightarrow{F}_C = \sup_{y \in C} D(\cdot, y) \tag{1.41}$$

is *convex* (as the supremum of convex functions) and then to apply the Ioffe-Tihomirov theorem (see, e.g., [40, Theorem 2.4.18]) for the subdifferential of the supremum of convex function. In contrast, $\overleftarrow{F}_C = \sup_{x \in C} D(x, \cdot)$ is generally *not convex*. (For more on separate and joint convexity of D, see [3].)

1.6 Chebyshev Centers: Two Examples

Diagonal-Symmetric Line Segments in the Strictly Positive Orthant

In addition to our standing assumptions from Section 1.1, we assume in this Subsection that the following hold:

$$X = \mathbb{R}^2; \tag{1.42}$$

$$\mathbf{c}_0 = (1, a) \text{ and } \mathbf{c}_1 = (a, 1), \quad \text{where } 1 < a < +\infty; \tag{1.43}$$

$$\mathbf{c}_\lambda = (1 - \lambda)\mathbf{c}_0 + \lambda \mathbf{c}_1, \quad \text{where } 0 < \lambda < 1; \tag{1.44}$$

$$C = \text{conv}\{\mathbf{c}_0, \mathbf{c}_1\} = \{\mathbf{c}_\lambda \mid 0 \le \lambda \le 1\}. \tag{1.45}$$

Theorem 1.6.1 *Suppose that f is any of the functions considered in Example 1.1.1. Then the left Chebyshev center is the midpoint of C, i.e., $\overleftarrow{Z}_C = \{\mathbf{c}_{1/2}\}$.*

Proof. By Theorem 1.5.2, we write $\overleftarrow{Z}_C = \{\mathbf{y}\}$, where $\mathbf{y} = (y_1, y_2) \in U$. In view of (1.37) and Fact 1.1.4(ii), we obtain that $\overleftarrow{Q}_C(\mathbf{y})$ contains at least two elements. On the other hand, since $\overleftarrow{Q}_C(\mathbf{y})$ consists of the maximizers of the *convex* function $D(\cdot, \mathbf{y})$ over the compact set C, [33, Corollary 32.3.2] implies that $\overleftarrow{Q}_C(\mathbf{y}) \subseteq \{\mathbf{c}_0, \mathbf{c}_1\}$. Altogether,

$$\overleftarrow{Q}_C(\mathbf{y}) = \{\mathbf{c}_0, \mathbf{c}_1\}. \tag{1.46}$$

In view of (1.37),

$$\mathbf{y} \in C. \tag{1.47}$$

On the other hand, a symmetry argument identical to the proof of [5, Proposition 5.1] and the uniqueness of its Chebyshev center show that $\mathbf{y}$ must lie on the diagonal, i.e., that

$$y_1 = y_2. \tag{1.48}$$

The result now follows because the only point satisfying both (1.47) and (1.48) is $\mathbf{c}_{1/2}$, the midpoint of C. ∎

Remark 1.6.2 Theorem 1.6.1 is in stark contrast with [5, Section 5], where we investigated the *right* Chebyshev center in this setting. Indeed, there we found that the right Chebyshev center does depend on the underlying Legendre function used (see [5, Examples 5.2, 5.3, and 5.5]). Furthermore, for each Legendre function f considered in Example 1.1.1, we obtain the following formula.

$$(\forall \mathbf{y} = (y_1, y_2) \in U) \quad \overleftarrow{Q}_{f,C}(\mathbf{y}) = \begin{cases} \{\mathbf{c}_0\}, & \text{if } y_2 < y_1; \\ \{\mathbf{c}_1\}, & \text{if } y_2 > y_1; \\ \{\mathbf{c}_0, \mathbf{c}_1\}, & \text{if } y_1 = y_2. \end{cases} \tag{1.49}$$

Indeed, since for every $\mathbf{y} \in U$, the function $D(\cdot, \mathbf{y})$ is convex; the points where the supremum is achieved is a subset of the extreme points of C, i.e., of $\{\mathbf{c}_0, \mathbf{c}_1\}$. Therefore, it suffices to compare $D(\mathbf{c}_0, \mathbf{y})$ and $D(\mathbf{c}_1, \mathbf{y})$.

Intervals of Real Numbers

Theorem 1.6.3 *Suppose that $X = \mathbb{R}$ and that $C = [a,b] \subset U$, where $a \neq b$. Denote the right and left Chebyshev centers by x and y, respectively. Then*[7]

$$x = \frac{f^*(b^*) - f^*(a^*)}{b^* - a^*} \quad and \quad y^* = \frac{f(b) - f(a)}{b - a}. \tag{1.50}$$

Proof. Analogously to the derivation of (1.46), it must hold that

$$\overleftarrow{Q}_C(y) = \{a,b\}. \tag{1.51}$$

This implies that y satisfies $D(a,y) = D(b,y)$. In turn, using Fact 1.1.4(i), this last equation is equivalent to $D_{f^*}(y^*,a^*) = D_{f^*}(y^*,b^*) \Leftrightarrow f^*(y^*) + f(a) - y^*a = f^*(y^*) + f(b) - y^*b \Leftrightarrow f(b) - f(a) = y^*(b-a) \Leftrightarrow y^* = (f(b) - f(a))/(b-a)$, as claimed. Hence

$$y = \nabla f^* \left(\frac{f(b) - f(a)}{b - a} \right). \tag{1.52}$$

Combining this formula (applied to f^* and $C^* = [a^*,b^*]$) with Lemma 1.2.3(ii), we obtain that the right Chebyshev center is given by

$$x = \nabla f^* \left(\nabla f^{**} \left(\frac{f^*(b^*) - f^*(a^*)}{b^* - a^*} \right) \right) = \frac{f^*(b^*) - f^*(a^*)}{b^* - a^*}, \tag{1.53}$$

as required.   ∎

Example 1.6.4 Suppose that $X = \mathbb{R}$ and that $C = [a,b]$, where $0 < a < b < +\infty$. In each of the following items, suppose that f is as in the corresponding item of Example 1.1.1. Denote the corresponding right and left Chebyshev centers by x and y, respectively. Then the following hold.

(i) $x = y = \dfrac{a+b}{2}$.

(ii) $x = \dfrac{b-a}{\ln(b) - \ln(a)}$ and $y = \exp\left(\dfrac{b\ln(b) - b - a\ln(a) + a}{b - a} \right)$.

(iii) $x = \dfrac{ab\left(\ln(b) - \ln(a)\right)}{b - a}$ and $y = \dfrac{b - a}{\ln(b) - \ln(a)}$.

Proof. This follows from Theorem 1.6.3.   ∎

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[7] Recall the convenient notation introduced on page 3!

1.7 Generalizations and Variants

Chebyshev set and Klee set problems can be generalized to problems involving functions.

Throughout this section,

$$g\colon X \to {]-\infty, +\infty]} \text{ is lower semicontinuous and proper.} \qquad (1.54)$$

For convenience, we also set

$$q = \tfrac{1}{2}\|\cdot\|^2. \qquad (1.55)$$

Recall that the *Moreau envelope* $e_\lambda g\colon X \to [-\infty, +\infty]$ and the set-valued *proximal mapping* $P_\lambda g\colon X \rightrightarrows X$ are given by

$$x \mapsto e_\lambda g(x) = \inf_w \left(g(w) + \frac{1}{2\lambda}\|x - w\|^2 \right) \qquad (1.56)$$

and

$$x \mapsto P_\lambda g(x) = \operatorname*{argmin}_w \left(g(w) + \frac{1}{2\lambda}\|x - w\|^2 \right). \qquad (1.57)$$

It is natural to ask: If $P_\lambda g$ is single-valued everywhere on $\mathbb{R}^n$, what can we say about the function g?

Similarly, define $\phi_\mu g\colon X \to {]-\infty, +\infty]}$ and $Q_\mu g\colon X \rightrightarrows X$ by

$$y \mapsto \phi_\mu g(y) = \sup_x \left(\frac{1}{2\mu}\|y - x\|^2 - g(x) \right), \qquad (1.58)$$

and

$$y \mapsto Q_\mu g(y) = \operatorname*{argmax}_x \left(\frac{1}{2\mu}\|y - x\|^2 - g(x) \right). \qquad (1.59)$$

Again, it is natural to ask: If $Q_\mu g$ is single-valued everywhere on X, what can we say about the function g? When $g = \iota_C$, then $P_\lambda g = P_C, Q_\mu g = Q_C$, and we recover the classical Chebyshev and Klee set problems.

Definition 1.7.1

(i) *The function g is* prox-bounded *if there exists $\lambda > 0$ such that $e_\lambda g \not\equiv -\infty$. The supremum of the set of all such λ is the threshold λ_g of the prox-boundedness for g.*

(ii) *The constant μ_g is defined to be the infimum of all $\mu > 0$ such that $g - \mu^{-1}q$ is bounded below on X; equivalently, $\phi_\mu g(0) < +\infty$.*

Fact 1.7.2 (See [34, Example 5.23, Example 10.32].) *Suppose that g is prox-bounded with threshold λ_g, and let $\lambda \in \;]0, \lambda_g[$. Then $P_\lambda g$ is everywhere upper semi-continuous and locally bounded on X, and $e_\lambda g$ is locally Lipschitz on X.*

Fact 1.7.3 (See [38, Proposition 4.3].) *Suppose that $\mu > \mu_g$. Then $Q_\mu g$ is upper semicontinuous and locally bounded on X, and $\phi_\mu g$ is locally Lipschitz on X.*

Definition 1.7.4

  (i) *We say that g is λ-Chebyshev if $P_\lambda g$ is single-valued on X.*
 (ii) *We say that g is μ-Klee if $Q_\mu g$ is single-valued on X.*

   Facts 1.7.5 and 1.7.7 below concern Chebyshev functions and Klee functions; see [38] for proofs.

Fact 1.7.5 (single-valued proximal mappings) *Suppose that g is prox-bounded with threshold λ_g, and let $\lambda \in \;]0, \lambda_g[$. Then the following are equivalent.*

  (i) *$e_\lambda g$ is continuously differentiable on X.*

 (ii) *g is λ-Chebyshev, i.e., $P_\lambda g$ is single-valued everywhere.*

(iii) *$g + \lambda^{-1}q$ is essentially strictly convex.*

If any of these conditions holds, then

$$\nabla\big((g + \lambda^{-1}q)^*\big) = P_\lambda g \circ (\lambda \,\mathrm{Id}). \tag{1.60}$$

Corollary 1.7.6 *The function g is convex if and only if $\lambda_g = +\infty$ and $P_\lambda g$ is single-valued on X for every $\lambda > 0$.*

Fact 1.7.7 (single-valued farthest mappings) *Suppose that $\mu > \mu_g$. Then the following are equivalent.*

  (i) *$\phi_\mu g$ is (continuously) differentiable on X.*

 (ii) *g is μ-Klee, i.e., $Q_\mu g$ is single-valued everywhere.*

(iii) *$g - \mu^{-1}q$ is essentially strictly convex.*

If any of these conditions holds, then

$$\nabla\big((g - \mu^{-1}q)^*\big) = Q_\mu g(-\mu \,\mathrm{Id}). \tag{1.61}$$

Corollary 1.7.8 *Suppose that g has bounded domain. Then $\mathrm{dom}\,g$ is a singleton if and only if for all $\mu > 0$, the farthest operator $Q_\mu g$ is single-valued on X.*

Definition 1.7.9 (Chebyshev points) *The set of μ-Chebyshev points of g is $\mathrm{argmin}\,\phi_\mu g$. If $\mathrm{argmin}\,\phi_\mu g$ is a singleton, then we denote its unique element by p_μ and we refer to p_μ as the μ-Chebyshev point of g.*

The following result is new.

Theorem 1.7.10 (Chebyshev point of a function) *Suppose that $\mu > \mu_g$. Then the set of μ-Chebyshev points is a singleton, and the μ-Chebyshev point is characterized by*

$$p_\mu \in \operatorname{conv} Q_\mu g(p_\mu). \tag{1.62}$$

Proof. As $\mu > \mu_g$, Fact 1.7.3 implies that

$$y \mapsto \phi_\mu g(y) = \frac{1}{2\mu}\|y\|^2 + \left(-\frac{1}{\mu}q + g\right)^*(-y/\mu), \tag{1.63}$$

is finite. Hence $\phi_\mu g$ is strictly convex and supercoercive; thus, $\phi_\mu g$ has a unique minimizer. Furthermore, we have

$$\partial \phi_\mu g(y) = \frac{1}{\mu}\big(y - \operatorname{conv} Q_\mu g(y)\big) \tag{1.64}$$

by the Ioffe-Tikhomirov Theorem [40, Theorem 2.4.18]. Therefore,

$$0 \in \partial \phi_\mu g(y) \quad \Leftrightarrow \quad y \in \operatorname{conv} Q_\mu g(y), \tag{1.65}$$

which yields the result.   ∎

We now provide three examples to illustrate the Chebyshev point of functions.

Example 1.7.11 Suppose that $g = q$. Then $\mu_g = 1$ and for $\mu > 1$, we have

$$\phi_\mu g \colon y \mapsto \sup_x \left(\frac{1}{2\mu}(y-x)^2 - \frac{x^2}{2}\right) = \frac{y^2}{2(\mu-1)}. \tag{1.66}$$

Hence the μ-Chebyshev point of g is $p_\mu = 0$.

Example 1.7.12 Suppose that $g = \iota_{[a,b]}$, where $a < b$. Then $\mu_g = 0$ and for $\mu > 0$, we have

$$\phi_\mu g \colon y \mapsto \sup_x \left(\frac{1}{2\mu}(y-x)^2 - \iota_{[a,b]}(x)\right) = \begin{cases} \frac{(y-b)^2}{2\mu} & \text{if } y \le \frac{a+b}{2}, \\ \frac{(y-a)^2}{2\mu} & \text{if } y > \frac{a+b}{2}. \end{cases} \tag{1.67}$$

Hence $p_\mu = \frac{a+b}{2}$.

Example 1.7.13 Let $a < b$ and suppose that g is given by

$$x \mapsto \begin{cases} 0 & \text{if } a \le x \le \frac{a+b}{2}, \\ 1 & \text{if } \frac{a+b}{2} < x \le b, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.68}$$

Then $\mu_g = 0$, and when $\mu > 0$ we have

$$\phi_\mu g(y) = \sup_x \left(\frac{1}{2\mu}(y-x)^2 - g(x) \right)$$

$$= \sup_x \begin{cases} \frac{1}{2\mu}(y-x)^2 & \text{if } a \leq x \leq \frac{a+b}{2} \\ \frac{1}{2\mu}(y-x)^2 - 1 & \text{if } \frac{a+b}{2} < x \leq b \\ -\infty & \text{otherwise} \end{cases}$$

$$= \max \left\{ \frac{(y-a)^2}{2\mu}, \frac{(y-(a+b)/2)^2}{2\mu}, \frac{(y-b)^2}{2\mu} - 1 \right\},$$

by using the fact that a strictly convex function only achieves its maximum at the extreme points of its domain. Elementary yet tedious calculations yield the following. When $\mu > (a-b)^2/4$, we have

$$\phi_\mu g(y) = \begin{cases} \frac{(y-b)^2}{2\mu} - 1 & \text{if } y < \frac{2\mu}{a-b} + \frac{a+3b}{4} \\ \frac{(y-(a+b)/2)^2}{2\mu} & \text{if } \frac{2\mu}{a-b} + \frac{a+3b}{4} \leq y < \frac{3a+b}{4} \\ \frac{(y-a)^2}{2\mu} & \text{if } y > \frac{3a+b}{4}; \end{cases}$$

while when $0 < \mu \leq (a-b)^2/4$, one obtains

$$\phi_\mu g(y) = \begin{cases} \frac{(y-b)^2}{2\mu} - 1 & \text{if } y < \frac{\mu}{a-b} + \frac{a+b}{2} \\ \frac{(y-a)^2}{2\mu} & \text{if } y \geq \frac{\mu}{a-b} + \frac{a+b}{2}. \end{cases}$$

Hence, the Chebyshev point of g is

$$p_\mu = \begin{cases} \frac{3a+b}{4}, & \text{if } \mu > (a-b)^2/4; \\ \frac{\mu}{a-b} + \frac{a+b}{2}, & \text{if } 0 < \mu \leq (a-b)^2/4. \end{cases}$$

1.8 List of Open Problems

Problem 1. Is the assumption that f be supercoercive in Fact 1.3.1 really essential?

Problem 2. Are the assumptions that f have full domain and that $\overline{C}^* \subseteq U^*$ in Fact 1.3.2 really essential?

Problem 3. Does there exist a Legendre function f with full domain such that f is not quadratic yet every nonempty closed convex subset of X is $\overrightarrow{D}$-Chebyshev? In view of Fact 1.3.1, the gradient operator ∇f of such a function would be nonaffine and it would preserve convexity.

Problem 4. Is it possible to characterize the class of $\overrightarrow{D}$-Chebyshev subsets of the strictly positive orthant when f is the negative entropy? Fact 1.3.4

and Example 1.3.3 imply that this class contains not only all closed convex but also some nonconvex subsets.

Problem 5.   Is the assumption that f be supercoercive in Fact 1.3.5 really essential?

Problem 6.   Is the assumption that f be supercoercive in Fact 1.4.4 really essential?

Problem 7.   For the Chebyshev functions and Klee functions, we have used the halved Euclidean distance. What are characterizations of f and Chebyshev point of f when one uses the Bregman distances?

Problem 8.   How do the results on Chebyshev functions and Klee functions extend to Hilbert spaces or even general Banach spaces?

1.9 Conclusion

Chebyshev sets, Klee sets, and Chebyshev centers are well known notions in classical Euclidean geometry. These notions have been studied traditionally also in an infinite-dimensional setting or with respect to metric distances induced by different norms. Recently, a new framework was provided by measuring the discrepancy between points differently, namely by Bregman distances, and new results have been obtained that generalize the classical results formulated in Euclidean spaces. These results are fairly well understood for Klee sets and Chebyshev centers with respect to Bregman distances; however, the situation is much less clear for Chebyshev sets.

The current state of the art is reviewed in this paper and several new results have been presented. The authors hope that the list of open problems (in Section 1.8) will entice the reader to make further progress on this fascinating topic.

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