INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Apr 16th, 2016 Location: ASC 140 Time: 1pm Duration 3 hours.
This exam has 8 questions for a total of 54 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Problem Number	1	2	3	4	5	6	7	8	**Total**
Points Earned									
Points Out Of	3	10	2	4	3	12	10	10	54

CANDIDATE NAME (print): ______________________

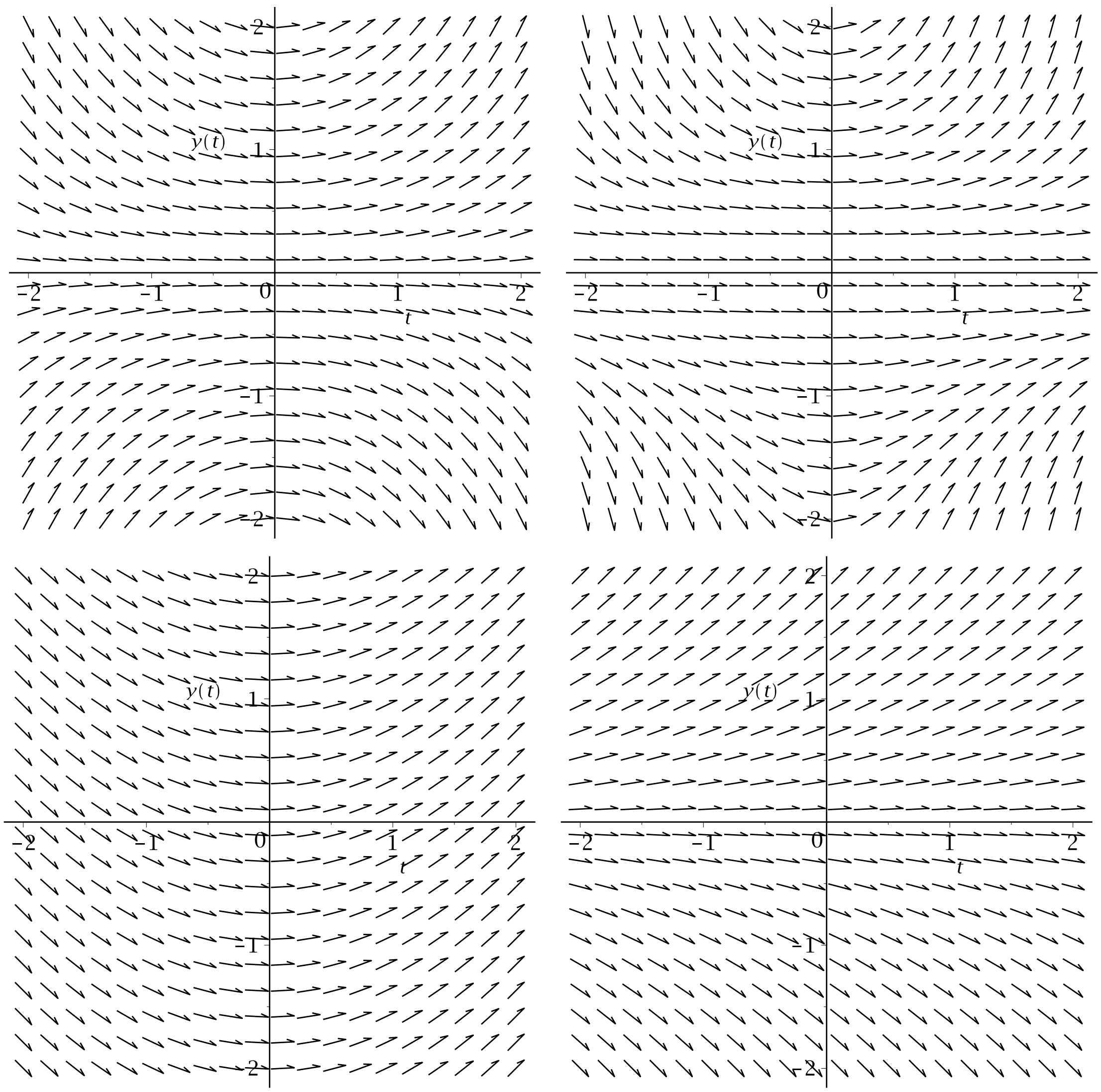
STUDENT NUMBER: ______________________

Signature: ______________________

[3] 1. Consider the ODE

$$\frac{dy}{dt} = kty^2, \qquad k > 0.$$

Which of the direction fields corresponds to the given ODE? Circle it. Then explain your choice.



[10] 2. A tank, of volume 1000 L, is initially filled with $x(0) = 0$ kg of salt. At time $t = 0$, inflow and outflow valves are opened. The inflow valve carries a brine solution containing $0.4\sin(t)$ kg of salt per litre. The inflow and outflow rates are both 10 L/min. Let $x(t)$ be the amount of salt in the tank at time t. Assuming the solution is kept well-stirred, determine the amount of salt in the tank at all times $t > 0$. Include a sketch in your solution. (Answers and work should be in **fractions**, not decimals.)

Hint: If you come across an integral that you cannot remember how to evaluate, you may find the integral you need in the table at the end of the exam. Note that you will lose a few marks if you simply use the result in the table and skip the integral evaluation step.

Workspace for problem #2.

2 3. Give the first four terms of the Taylor series expansion of $f(t) = \sin(at)$ about $t = 0$. The parameter a is an arbitrary constant.

4. Consider the initial value problem

$$\frac{du}{dt} = 2u, \qquad u(0) = 1. \tag{1}$$

1 (a) Write the Forward Euler formula for u_{n+1} corresponding to the given ODE.

3 (b) Using the Forward Euler method, and $h = \Delta\, t = 0.1$, fill in the table below. Show your work!

n	u_n	u_{n+1}
0	1	
1		

3 5. The existence and uniqueness theorem for variable coefficient equations is as follows:

Theorem 5: Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists on the same interval (a, b) a unique solution $y(t)$ to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

Use Theorem 5 to discuss the existence and uniqueness of solutions to the IVPs below:

(a) $t(t-3)y'' + 2ty' - y = t^2, \qquad y(1) = -1, \quad y'(1) = 2$

(b) $t^2z'' + tz' + z = \cos(t), \qquad z(0) = 1, \quad z'(0) = 0$

12 6. Determine the equation of motion for a mass-spring system governed by

$$\frac{d^2y}{dt^2} + y = 5\cos(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

Sketch the solution and name the behaviour.

Workspace for problem #6.

10 7. Find the general solution of the ODE

$$\frac{1}{2}y'' + 2y = \frac{1}{\cos(2t)},$$

using variation of parameters. Use fractions, not decimals, in your work and answer.

10 8. Using the method of Laplace transforms, solve the initial value problem

$$y'' + 5y' + 6y = e^{-t}\delta(t-2), \qquad y(0) = 0, \quad y'(0) = 0.$$

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$\delta(t-a)$	$e^{-as}, \quad s > 0$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s-a)$$
$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)H(t-a),$$

where $H(t)$ is the Heaviside function.

A POSSIBLY USEFUL INTEGRAL:

$$\int e^{bx}\sin(ax)dx = \frac{1}{a^2+b^2}e^{bx}\left(b\sin(ax) - a\cos(ax)\right) + C$$