INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

Irving K. Barber School
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UBC Okanagan

Date: Apr 16th, 2016 Location: ASC 140 Time: 1pm Duration 3 hours.
This exam has 8 questions for a total of 54 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Problem Number	1	2	3	4	5	6	7	8	Total
Points Earned									
Points Out Of	3	10	2	4	3	12	10	10	54

CANDIDATE NAME (print): Solutions

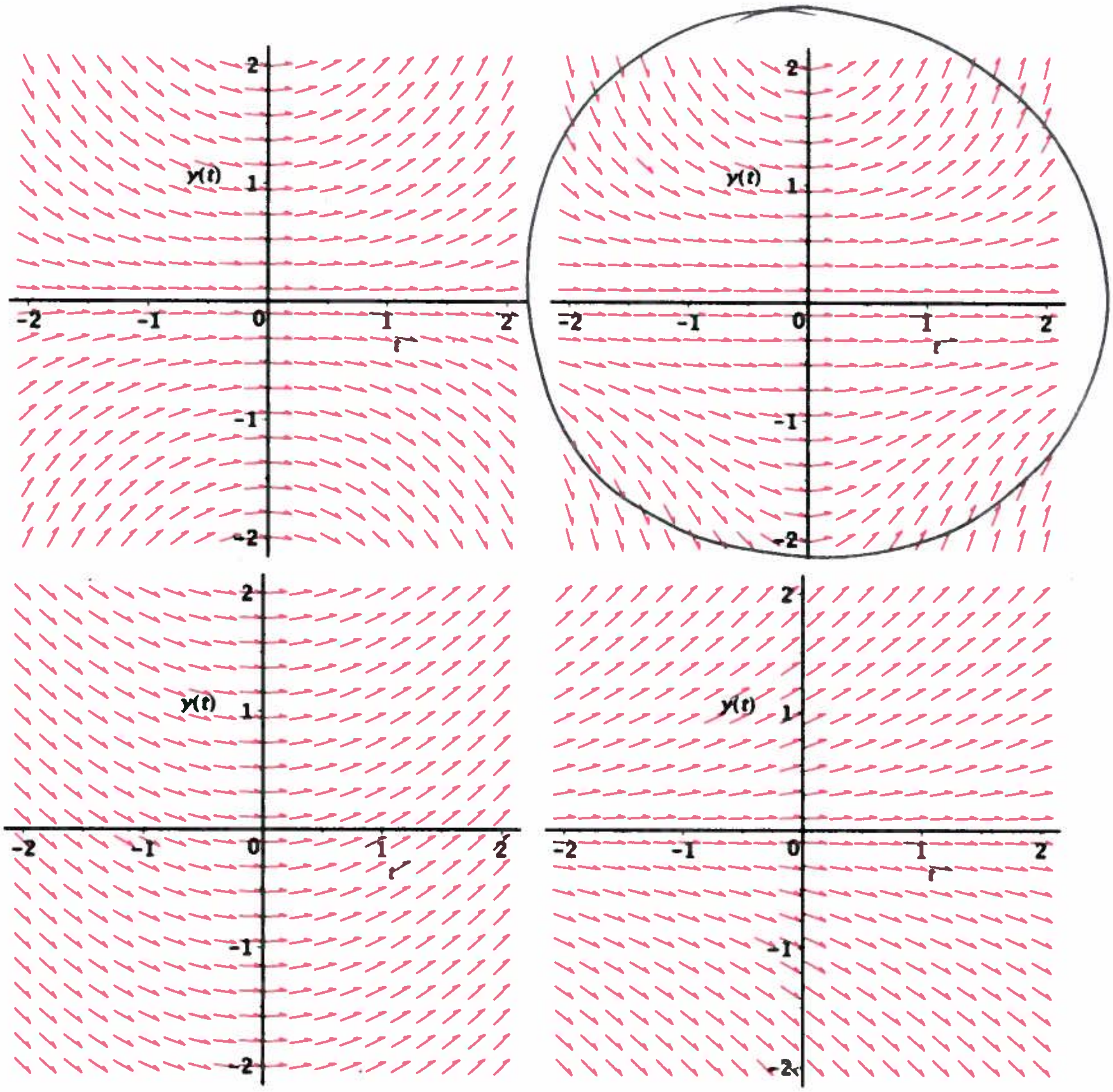
STUDENT NUMBER: ______

Signature: ______

$\boxed{3}$ 1. Consider the ODE

$$\frac{dy}{dt} = kty^2, \qquad k > 0.$$

Which of the direction fields corresponds to the given ODE? Circle it. Then explain your choice.



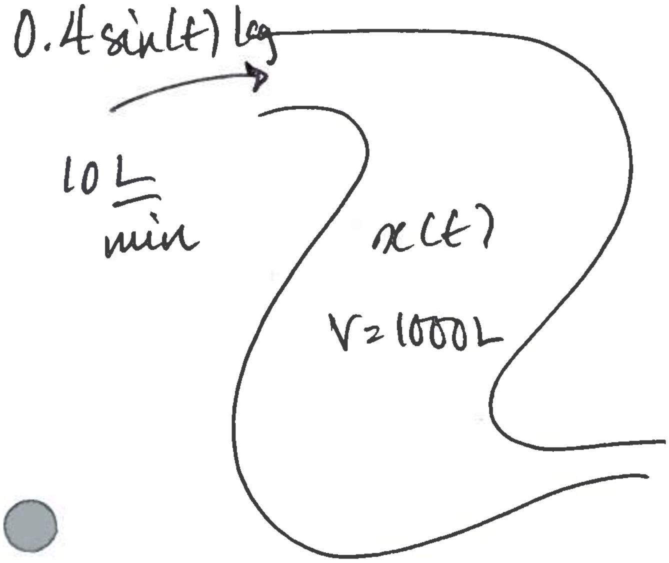
$$kty^2 \begin{cases} > 0 & \forall\, t > 0 \\ < 0 & \forall\, t < 0 \end{cases}$$

$\therefore$ the 1st + 4th plots can be eliminated

$kty^2 = 0 \Rightarrow \{ y = 0$ OR $t = 0$

$\therefore$ the 3rd plot can be eliminated (not zero slope on $y=0$)

$\boxed{10}$ 2. A tank, of volume 1000 L, is initially filled with $x(0) = 0$ kg of salt. At time $t = 0$, inflow and outflow valves are opened. The inflow valve carries a brine solution containing $0.4\sin(t)$ kg of salt per litre. The inflow and outflow rates are both 10 L/min. Let $x(t)$ be the amount of salt in the tank at time t. Assuming the solution is kept wll-stirred, determine the amount of salt in the tank at all times $t > 0$. Include a sketch in your solution. (Answers + work should be in fractions, not decimals)



$$\frac{dx}{dt} = 0.4\sin(t)\cdot 10 - \frac{x}{1000}\cdot 10 \iff (I)$$

$$(I) \iff x' + \frac{x}{100} = 4\sin(t)$$

We multiply the ODE by an integrating factor:

$$\frac{d}{dt}\left(e^{\frac{1}{100}t}x\right) = 4e^{\frac{1}{100}t}\sin(t) \iff (II)$$

$$(II) \iff e^{\frac{1}{100}t}x = 4\underbrace{\int e^{\frac{1}{100}t}\sin(t)\,dt}_{I}$$

To solve the integral I, we use integration by parts.

Workspace for problem #2.

$$I = \int e^{\frac{1}{100}t}\sin(t)\,dt \qquad \text{let } u=\sin(t) \quad du=\cos(t)\,dt$$
$$dv = e^{\frac{1}{100}t}\,dt \quad v=100e^{\frac{1}{100}t}$$

$$= 100e^{\frac{1}{100}t}\sin(t) - 100\int e^{\frac{1}{100}t}\cos(t)\,dt$$

$$\text{let } u=\cos(t) \quad du=-\sin(t)\,dt$$
$$dv = e^{\frac{1}{100}t}\,dt \quad v=100e^{\frac{1}{100}t}$$

$$= 100e^{\frac{1}{100}t}\sin(t) - 100\left[100e^{\frac{1}{100}t}\cos(t) + 100I\right]$$

Solving for I we obtain:

$$\frac{I\,(1+100^2)}{(1+100^2)} = \frac{100e^{\frac{1}{100}t}}{(1+100^2)}\left(\sin(t) - 100\cos(t)\right)$$

Substituting this result into the equation for $x(t)$:

$$x(t) = (4I + C)\,e^{-\frac{1}{100}t}$$

$$= \frac{400}{10001}\left(\sin(t) - 100\cos(t)\right) + Ce^{-\frac{1}{100}t}$$

Applying the ICs: $x(0)=0 \Rightarrow -\frac{40000}{10001} + C = 0 \Rightarrow C = \frac{40000}{10001}$

$$\therefore \boxed{x(t) = \frac{400}{10001}\left[\sin(t) - 100\cos(t) + 100e^{-\frac{1}{100}t}\right]}$$

1st 4 terms of the

[2] 3. Give the Taylor series expansion of $f(t) = \sin(at)$ about $x = 0$. The parameter a is an arbitrary constant.

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \frac{f'''(0)}{3!}t^3 + \dots$$

$$= 0 + at + 0 - \frac{a^3t^3}{3!} + \dots$$

4. Consider the initial value problem

$$\frac{du}{dt} = 2u, \qquad u(0) = 1. \tag{1}$$

[1] (a) Write the Forward Euler formula for y_{n+1} corresponding to the given ODE.

$$u_{n+1} = u_n + h\,2u_n$$

[3] (b) Using the Forward Euler method, and $h = \Delta\, t = 0.1$, fill in the table below. Show your work!

n	u_n	u_{n+1}
0	1	$u_1 = 1 + 0.1(2\cdot 1) = 1 + 0.1(2)$ $= 1.2$
1	1.2	$u_2 = 1.2 + 0.1(2\cdot(1.2)) = 1.2 + 0.1(2.4)$ $= 1.2 + 0.24 = 1.44$

$\boxed{3}$ 5. The existence and uniqueness theorem for variable coefficient equations is as follows:

> **Theorem 5:** Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists on the same interval (a, b) a unique solution $y(t)$ to the initial value problem
>
> $$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

Use Theorem 5 to discuss the existence and uniqueness of a solution to the two IVPs below:

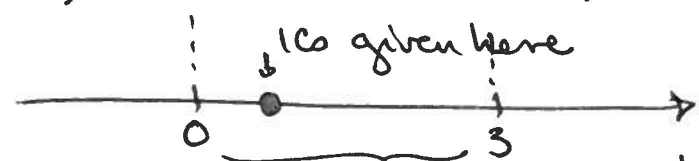
$$\text{(a)}t(t-3)y'' + 2ty' - y = t^2, \qquad y(1) = -1, \quad y'(1) = 2$$

rearranging the ODE:

$$y'' + \frac{2t}{t(t-3)}y' - \frac{1}{t(t-3)}y = \frac{t^2}{t(t-3)}$$

$p(t), q(t) + g(t)$: discontinuities at $t = 0$ & 3

$\therefore$



solution exists here: $t \in (0, 3)$

$$\text{(b)}t^2z'' + tz' + z = \cos(t), \qquad z(0) = 1, \quad z'(0) = 0$$

rearranging:

$$z'' + \frac{t}{t^2}z' + \frac{1}{t^2}z = \frac{\cos(t)}{t^2}$$

$p(t), q(t) + g(t)$: discontinuities at $t = 0$

$\because$ the ICs are given at the point of discontinuity, Theorem 5 does not apply.

12 6. Determine the equation of motion for a mass-spring system governed by

$$\frac{d^2y}{dt^2} + y = 5\cos(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

Sketch the solution and name the behaviour.

homogeneous solution:

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$

$\therefore y_h(t) = c_1 \cos(t) + c_2 \sin(t)$

particular solution:

$$\begin{cases} y_p(t) = At\cos(t) + Bt\sin(t) \\ y_p'(t) = A\cos(t) + B\sin(t) - At\sin(t) + Bt\cos(t) \\ y_p''(t) = -2A\sin(t) + 2B\cos(t) - At\cos(t) - Bt\sin(t) \end{cases}$$

plugging y_p into the ODE:

$y_p'' + y_p = 5\cos(t) \Rightarrow -2A\sin(t) + 2B\cos(t) = 5\cos(t)$

$\therefore A = 0$ & $B = \frac{5}{2}$

we obtain

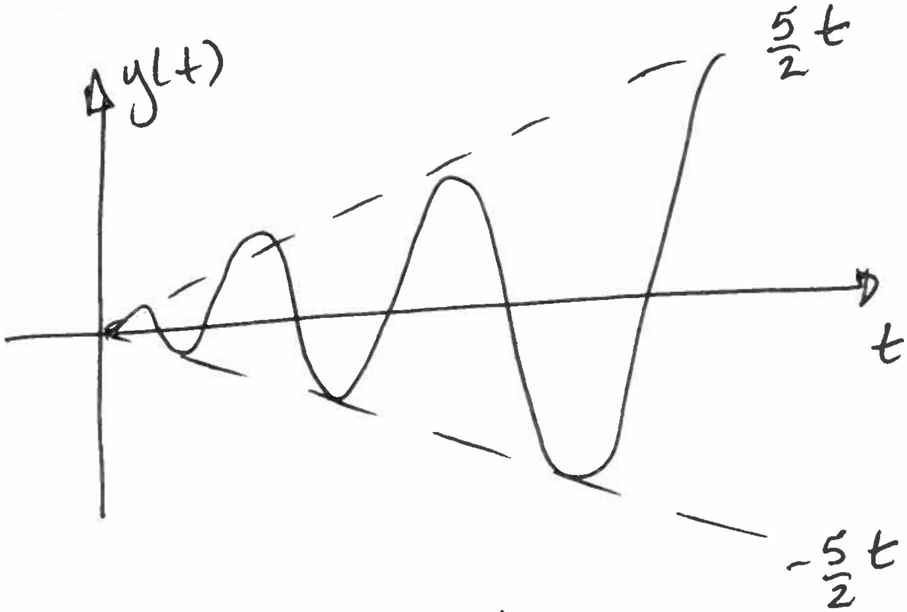
$y(t) = c_1\cos(t) + c_2\sin(t) + \frac{5}{2}t\sin(t)$

Workspace for problem #6.

We apply the ICs:

$$\begin{cases} y(0)=0 \\ y'(0)=1 \end{cases} \qquad \begin{cases} c_1 = 0 \\ c_2 = 1 \end{cases}$$

$$\therefore\ y(t) = \sin(t) + \frac{5}{2} t \sin(t)$$



The amplitude increases linearly (& without bound). This behaviour is called resonance.

$\boxed{10}$ 7. Find the general solution of the ODE

$$\frac{1}{2}y'' + 2y = \frac{1}{\cos(2t)}$$

Use fractions, not decimals, in your work and answer.

homogeneous solution

$$\frac{1}{2}r^2 + 2 = 0 \Leftrightarrow r^2 = -4 \Leftrightarrow r = \pm 2i$$

$$\therefore y_h = c_1 \cos(2t) + c_2 \sin(2t)$$

particular solution

$$y_p(t) = v_1 \cos(2t) + v_2 \sin(2t)$$

where $v_1(t)$ + $v_2(t)$ satisfy

$$\begin{cases} v_1' \cos(2t) + v_2' \sin(2t) = 0 \\ -2v_1' \sin(2t) + 2v_2' \cos(2t) = \dfrac{2}{\cos(2t)} \end{cases} \Leftrightarrow 1/$$

$$1/ \Leftrightarrow \begin{cases} 2v_1' = -2\tan(2t) \\ 2v_2' = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} v_1' = -\tan(2t) \\ v_2' = 1 \end{cases} \Leftrightarrow \begin{cases} v_1 = \frac{1}{2}\ln(\cos(2t)) + \cancel{c}^{0} \\ v_2 = t + \cancel{c}^{0} \end{cases}$$

$$\therefore y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{2}\ln(\cos(2t))\cos(2t) + t\sin(2t$$

$\boxed{10}$ 8. Using the method of Laplace transforms, solve the initial value problem

$$y'' + 5y' + 6y = e^{-t}\delta(t-2), \qquad y(0) = 0, \quad y'(0) = 0.$$

Taking the Laplace transform of both sides:

$$s^2 Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 6Y(s) = e^{-2s}e^{-2}$$

$$\Leftrightarrow (s^2 + 5s + 6)Y(s) = e^{-2}e^{-2s}$$

$$\Leftrightarrow Y(s) = \frac{e^{-2}}{(s+2)(s+3)} e^{-2s}$$

Let

$$F(s) = \frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{A(s+3) + B(s+2)}{(s+2)(s+3)}$$

$$\therefore \begin{cases} A + B = 0 \\ 3A + 2B = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$$

So $F(s) = \frac{1}{s+2} - \frac{1}{s+3}$ $\quad \therefore f(t) = e^{-2t} - e^{-3t}$

$$\therefore \mathcal{L}^{-1}\{e^{-2}F(s)e^{-2s}\} = e^{-2}\mathcal{L}^{-1}\{F(s)e^{-2s}\} = \mathrm{I}$$

$$\mathrm{I} = e^{-2}f(t-2)H(t-2) = e^{-2}\left(e^{-2(t-2)} - e^{-3(t-2)}\right)H(t-2)$$

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s-a)$$
$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)H(t-a),$$

where $H(t)$ is the Heaviside function.