INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

a place of mind UBC

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Feb 1st, 2015 Time: 11:30am Duration 35 minutes.
This exam has 6 questions for a total of 28 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Problem	Points Earned	Points Out Of
1		7
2		7
3		5
4		5
5		4
TOTAL:		**28**

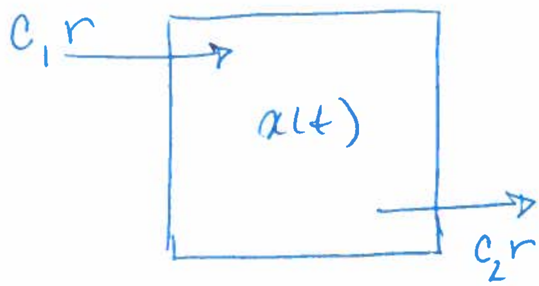
CANDIDATE NAME (print): Solutions

STUDENT NUMBER: ______

Signature: ______

$\boxed{7}$ 1. An auditorium is 100 m in length, 50 m in width, and 30 m in height. It is ventilated by a system that feeds in fresh air and draws out air at the same rate. If the auditorium air is well-mixed, what inflow (and outflow) rate is required to reduce any air pollutants present by a factor of 100 in 30 minutes? Include a diagram in your solution.

$V = 100 \times 50 \times 30 = 150\,000\ m^3$



$x(t)$ = amt of pollutant @ time t

$c_1 = 0$ (fresh air)

$c_2 = \frac{x(t)}{V}$

$\frac{dx}{dt} = 0 \cdot r - \frac{x}{V} r \Leftrightarrow \frac{dx}{dt} = -\frac{x}{V} r \Leftrightarrow \int \frac{dx}{x} = \int -\frac{r}{V} dt$

$\Leftrightarrow \ln|x| = -\frac{r}{V} t + C$

$\Leftrightarrow x = x_0 e^{-\frac{r}{V}t}$ where $x_0 = x(0)$

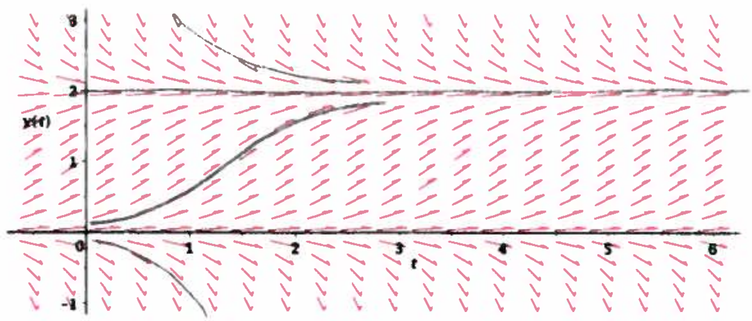
Require

$\frac{x(0)}{x(30)} = 100 \Leftrightarrow \frac{x_0 e^{-\frac{r}{V}\cdot 0}}{x_0 e^{-\frac{r}{V}30}} = 100 \Leftrightarrow e^{\frac{r}{V}30} = 100$

$\Leftrightarrow \frac{r}{V} \cdot 30 = \ln(100) \Leftrightarrow r = \frac{V \ln(100)}{30}$

$\Leftrightarrow r = \frac{150\,000 \ln(100)}{30} = 5000 \ln(100)$

2. Use the given direction field to answer the questions below.



$\boxed{2}$ (a) Draw several solution curves, some starting at $y = 2$ some starting at y just above zero, and some starting at y just below zero.

$\boxed{3}$ (b) What type of ODE produced this direction field? Write a plausible guess for what this ODE is;

This direction field was produced by an autonomous ODE as the isoclines are horizontal lines.

$$\frac{dy}{dt} = y(2-y)$$

$\boxed{2}$ (c) What can you say about the solution trajectories as $t \to -\infty$?

as $t \to -\infty$, all solutions with

$y(0) = 2 \quad \to 2$

$y(0) < 2 \quad \to 0$

$y(0) > 2 \quad \to +\infty$

[5] 3. Find the solution to the initial value problem

$$y' + 2ty = 2te^{-t^2}, \qquad y(1) = 2.$$

integrating factor:

$$\mu(t) = e^{\int p(t)dt} = e^{\int 2t\,dt} = e^{t^2}$$

multiply the ODE by $\mu(t)$.

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 2t \Leftrightarrow 1/$$

$$1/ \Leftrightarrow \frac{d}{dt}\left(e^{t^2}y\right) = 2t \Leftrightarrow e^{t^2}y = t^2 + C$$

$$\Leftrightarrow y = (t^2 + C)e^{-t^2}$$

apply the IC:

$$y(1) = 2 \Leftrightarrow 2 = (1+C)e^{-1} \Leftrightarrow 2e = 1 + C$$

$$\Leftrightarrow C = 2e - 1$$

$$\therefore \boxed{y(t) = (t^2 + 2e - 1)e^{-t^2}}$$

[5] 4. Find an integrating factor for the equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0.$$

Make sure you verify that your new equation is indeed exact! (*Hint: The integrating factor is a function of x.*)

Let $M(x,y) = 3xy + y^2$, $N(x,y) = x^2 + xy$.

Then

$$\frac{\partial M}{\partial y} = 3x + 2y \neq \frac{\partial N}{\partial x} = 2x + y$$

So the ODE is not exact. We look for an integrating factor that is a function of x:

$$\frac{\partial}{\partial y}(\mu M) = \mu \frac{\partial M}{\partial y} = \mu(3x + 2y) \quad \ldots\ldots (2a)$$

$$\frac{\partial}{\partial x}(\mu N) = \mu \frac{\partial N}{\partial x} + N\mu' = \mu(2x+y) + \mu'(x^2+xy) \ldots (2b)$$

Setting (2a) = (2b) we have

$$\mu(3x + 2y) = \mu(2x+y) + \mu'(x^2+xy) \Leftrightarrow$$

$$\Leftrightarrow \frac{\mu'}{\mu} = \frac{+(x+y)}{x^2+xy} = \frac{+(x+y)}{x(x+y)} = +\frac{1}{x}$$

$$\Leftrightarrow \int \frac{d\mu}{\mu} = \int -\frac{1}{x}\,dx \Leftrightarrow \ln|\mu| = +\ln|x|$$

$$\Leftrightarrow \boxed{\mu = x}$$

$\boxed{4}$ 5. Use the Forward Euler method to approximate the solution to the IVP below using steps of size $h = 0.1$. Enter your results in the table provided. Show your calculations below.

$$\frac{dv}{dt} = \frac{t}{v}, \qquad v(0) = -1.$$

n	t_n	v_n	v_{n+1} $= v_n + h\, t_n / v_n$
0	0	-1	-1
1	0.1	-1	-1.01

F.E. Method:

$$v_{n+1} = v_n + h \frac{t_n}{v_n}$$

$$v_1 = v_0 + h\frac{t_0}{v_0} = -1 + (0.1)\frac{0}{-1} = -1$$

$$v_2 = v_1 + h\frac{t_1}{v_1} = -1 + (0.1)\frac{(0.1)}{(-1)} = -1 - 0.01$$

$$= -1.01$$

6. **BONUS PROBLEM for the Group Test** (you can start working on this problem while waiting for the group test to start): Solve (implicitly) the exact differential equation

$$\underbrace{(\sin(x) + x^2e^y - 1)}_{N(x,y)}dy + \underbrace{(y\cos(x) + 2xe^y)}_{M(x,y)}dx = 0.$$

$$\frac{\partial M}{\partial y} = \cos(x) + 2xe^y = \frac{\partial N}{\partial x} = \cos(x) + 2xe^y$$

$\therefore$ the ODE is exact. We $\therefore$ seek F such that

$$F = \int (y\cos(x) + 2xe^y)\,dx = y\sin(x) + x^2e^y + h(y)$$

and then

$$\frac{\partial F}{\partial y} = N \iff \sin(x) + x^2e^y + h'(y) = \sin(x) + x^2e^y - 1$$

$$\iff h'(y) = -1 \iff h(y) = -y + K$$

$\therefore$ $F(x,y) = y\sin(x) + x^2e^y - y$, and the solutions of the ODE are the level curves of F:

$$\boxed{y\sin(x) + x^2e^y - y = C}$$