INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Feb 3rd, 2016 Time: 11:30am Duration 35 minutes.
This exam has 5 questions for a total of 20 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Problem	Points Earned	Points Out Of
1		6
2		6
3		5
4		3
TOTAL:		**20**

CANDIDATE NAME (print): Solutions

STUDENT NUMBER:

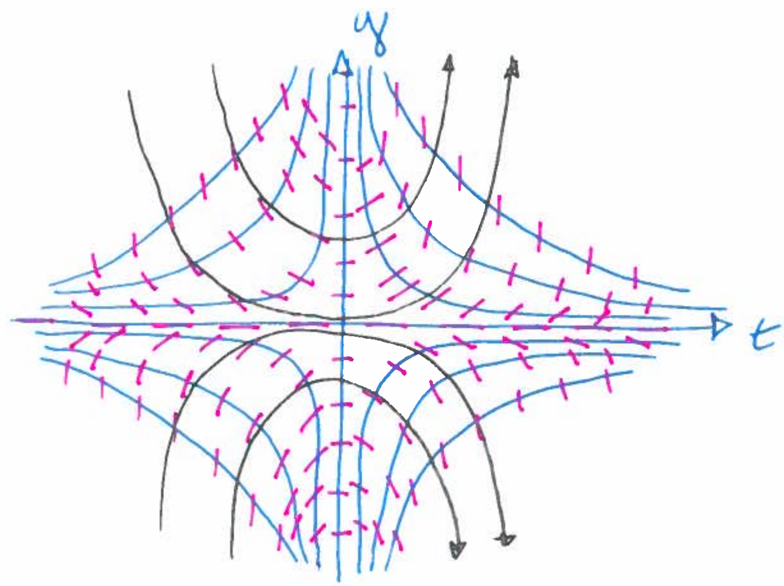
Signature:

$\boxed{6}$ 1. Consider the ODE

$$\frac{dy}{dt} = ty.$$

(a) Use the method of isoclines to determine the direction field for the ODE. Your plot should include several isoclines with direction field arrows, and a few representative solution curves.

isoclines: $\frac{dy}{dt} = C \iff ty = C \iff y = \frac{C}{t}$



• In quadrant I & III, isoclines for $C>0$

• In II & IV, isocline for $C<0$

• $C=0$ on the t & y axes

— = isoclines

— = slopes

— = solution curves

(b) Could you represent the behaviour of this ODE using a phase line? Why or why not?

No, because the ODE is <u>non</u>autonomous

6 2. Solve the initial value problem

$$\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x, \qquad y(1) = 1.$$

$\frac{dy}{dx} + \frac{3}{x}y = 3x - 2$, We seek an integrating factor:

$\mu(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln|x^3|}$

$= |x^3|$, take $\boxed{\mu(x) = x^3}$

multiplying the ODE by $\mu(x)$, we obtain

$x^3 \frac{dy}{dx} + 3x^2 y = 3x^4 - 2x^3 \Leftrightarrow \frac{d}{dx}(x^3 y) = 3x^4 - 2x^3 \Leftrightarrow$

$\Leftrightarrow x^3 y = \frac{3}{5}x^5 - \frac{2}{4}x^4 + C \Leftrightarrow \boxed{y = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{C}{x^3}}$

Apply the IC:

$y(1) = 1 \Leftrightarrow \frac{3}{5} - \frac{1}{2} + C = 1 \Leftrightarrow C = 1 - \frac{3}{5} + \frac{1}{2} = \frac{9}{10}$

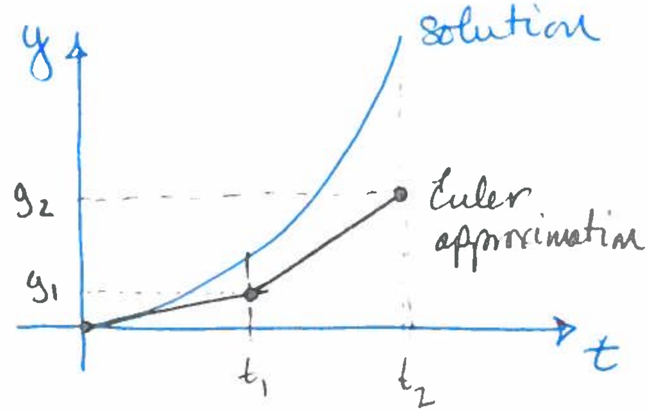
$\therefore \boxed{y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10x^3}}$

$\boxed{5}$ 3. Assume we are considering the direction field of an autonomous first order differential equation. Suppose we can qualitatively establish, by examining this direction field, that all solution curves $y(t)$ in a given region of the ty-plane exhibit one of the following two types of behaviour:

(i) increasing, concave up (ii) increasing, concave down

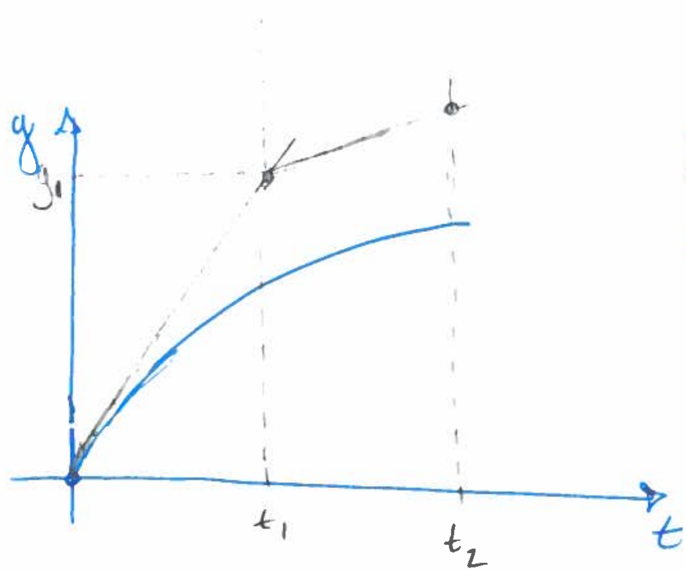
Suppose we implement a Forward Euler approximation to one of the solution curves in the region, using some reasonable step size h. Consider each of the ~~four~~ two cases. In each case, will the values y_h underestimate the exact values or oversetimate the exact values, or is it impossible to reach a definite conclusion? Explain your answer, and include sketches.

i)



∵ Euler's method takes the slope at the beginning of each t interval, y_h will underestimate the true value of y.

ii)



In this case, the slope decreases throughout each timestep, & so Euler's method overestimates the true value of y.

$\boxed{3}$ 4. Suppose that a good model for the population of invasive Eastern black and gray squirrels in Kelowna is given by

$$\frac{dN}{dt} = rN(K - N),$$

where N is the squirrel population at time t, and r and K are constant parameters.

(a) Sketch the phase line for the ODE.

(b) If the initial population of squirrels is $N(0) = N_0$, then the solution of the IVP is

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rKt}}.$$

Now suppose that at time $t = T$, the city decides that the invasive squirrels are becoming a problem, and starts harvesting the squirrels at constant rate H. What is the new IVP that applies for $t \geq T$?

$$N(T) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rKT}}$$

$\therefore$ the new IVP is

$$\frac{dN}{dt} = rN(K-N) - H, \quad t \geq T$$

$$N(T) = \frac{N_0 K}{N_0 + (K - N_0)e^{-rKT}}$$

5. **BONUS problem for the Group Test** Find the most gneral function $N(u,v)$ so that the equation below is exact:

$$(ve^{uv} - 4u^3v + 2)du + N(u,v)dv = 0$$

For exactness require:

$$\frac{\partial}{\partial v}\left(ve^{uv} - 4u^3v + 2\right) = \frac{\partial N}{\partial u} \iff //$$

$$// \iff e^{uv} + uve^{uv} - 4u^3 = \frac{\partial N}{\partial u}$$

$$\iff N = \int \left(e^{uv} + uve^{uv} - 4u^3\right) du$$

$w = uv$
$dw = v\,du$

$$= \frac{1}{v}e^{uv} - u^4 + h(v) + \int we^{w}\frac{dw}{v}$$

Solving the remaining integral:

$$\int we^{w}dw = we^{w} - \int e^{w}dw$$

$u = w$ $\quad du = dw$
$dv = e^{w}dw$ $\quad v = e^{w}$

$$= we^{w} - e^{w}$$

$$= e^{w}(w-1)$$

Plugging this result into the expression for N, we obtain

$$\boxed{\begin{aligned} N &= \frac{1}{v}e^{uv} - u^4 + \frac{e^{uv}}{v}(uv-1) + h(v) \\ &= ue^{uv} - u^4 + h(v) \end{aligned}}$$