INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Mar 16th, 2016 Time: 11:30am Duration 35 minutes.
This exam has 5 questions for a total of 22 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Problem	Points Earned	Points Out Of
1		6
2		6
3		5
4		3
TOTAL:		**22**

CANDIDATE NAME (print): Solutions

STUDENT NUMBER: ______

Signature: ______

$\boxed{3}$ 1. Determine the **form** of a particular solution to the ODE below. Do **not** solve for the coefficients!

$$y'' + 4y' + 5y = e^{2t} + t^2\sin(t) - e^{2t}\cos(t)$$

$$y_p(t) = Ae^{2t} + (B_1t^2 + C_1t + D_1)\sin(t) + (B_2t^2 + C_2t + D_2)\cos(t)$$
$$+ E_1te^{2t}\cos(t) + E_2te^{2t}\sin(t)$$

2. a mass weighing 8 kg is attached to a spring with stiffness constant 10 N/m. At $t = 0$, the mass is at it's equilibrium position, and an external force $F(t) = 2\cos(2t)$ N is applied to the system. The damping constant for the system is 1 Ns/m.

$\boxed{2}$ (a) Write the IVP that governs the system.

$$8y'' + y' + 10y = 2\cos(2t), \quad y(0) = y'(0) = 0$$

$\boxed{1}$ (b) What is the resonance frequency of the system? (give the **exact** answer)

$$\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{10}{8} - \frac{1}{128}} = \sqrt{\frac{159}{128}}$$
$$= \frac{1}{8}\sqrt{\frac{159}{2}}$$

$\boxed{7}$ 3. Consider the ODE $y'' + y = \tan(t)$. Given that $y_h(t) = c_1\cos(t) + c_2\sin(t)$, find a particular solution.

(Hint: You may find it useful to know that $\int \sec(u)du = \ln|\sec(u)\tan(u)| + C$.*)*

Suppose $y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$.

Then $v_1(t)$ + $v_2(t)$ must satisfy

$$\begin{cases} v_1'\cos(t) + v_2'\sin(t) = 0 \\ -v_1'\sin(t) + v_2'\cos(t) = \tan(t) \end{cases} \iff \text{ⁱ/}$$

$$\text{ⁱ/} \iff \begin{cases} v_2'(\sin^2(t) + \cos^2(t)) = \tan(t)\cos(t) \\ v_1'(\cos^2(t) + \sin^2(t)) = -\tan(t)\sin(t) \end{cases}$$

$$\iff \begin{cases} v_2' = \sin(t) \\ v_1' = \dfrac{-\sin^2(t)}{\cos(t)} = \dfrac{-(1-\cos^2(t))}{\cos(t)} = -\sec(t) + \cos(t) \end{cases}$$

$$\iff \begin{cases} v_2 = -\cos(t) + \cancel{c}^{\to 0} \\ v_1 = -\ln|\sec(t) + \tan(t)| + \sin(t) + \cancel{c}^{\to 0} \end{cases}$$

$$\therefore y_p(t) = [-\ln|\sec(t) + \tan(t)| + \sin(t)]\cos(t) + [-\cos(t)]\sin(t)$$

$$= -\cos(t)\ln|\sec(t) + \tan(t)|$$

9 4. Consider the following forced spring-mass system:

$$4y'' + 4y' + 5y = 17\cos(t) \quad y(0) = y'(0) = 0.$$

Given that

$$y_h(t) = e^{-\frac{1}{2}t}(c_1\cos(t) + c_2\sin(t)),$$

find the equation of motion of the mass.

We need to find a particular solution. We try

$$\begin{cases} y_p(t) = A\cos(t) + B\sin(t) \\ y_p'(t) = -A\sin(t) + B\cos(t) \\ y_p''(t) = -A\cos(t) - B\sin(t) \end{cases}$$

Plugging y_p, y_p', & y_p'' into the ODE we have

$$4y_p'' + 4y_p' + 5y_p = \cos(t) \Leftrightarrow (1)$$

$$(1) \Leftrightarrow -4A\cos(t) - 4B\sin(t) - 4A\sin(t) + 4B\cos(t) + 5A\cos(t) + 5B\sin(t) = 17\cos(t)$$

$$\Leftrightarrow \begin{cases} -4A + 4B + 5A = 17 \\ -4B - 4A + 5B = 0 \end{cases} \Leftrightarrow \begin{cases} A + 16A = 17 \\ B = 4A \end{cases}$$

$$\Leftrightarrow \begin{cases} A = 1 \\ B = 4 \end{cases}$$

$\therefore$ $y_p(t) = \cos(t) + 4\sin(t)$

and so the general solution is

$$y(t) = e^{-\frac{1}{2}t}\left(c_1\cos(t) + c_2\sin(t)\right) + \cos(t) + 4\sin(t)$$

Workspace for question 4.

Now we apply the ICs:

$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases} \iff \begin{cases} c_1+1=0 \\ -\frac{1}{2}c_1+c_2+4=0 \end{cases} \iff \begin{cases} c_1=-1 \\ c_2=-\frac{9}{2} \end{cases}$$

Thus, the equation of motion is

$$\boxed{y(t) = -e^{-\frac{1}{2}t}\left(\cos(t)+\frac{9}{2}\sin(t)\right) + \cos(t) + 4\sin(t)}$$

5. **BONUS problem for the Group Test** Use the mass-spring analogy to determine the qualitative form of solutions to the IVP

$$U'' + cU' + U(1-U) = 0, \qquad U(0) = 0.9, \quad U'(0) = 0,$$

where c is an arbitary positive constant. *(Note: This equation is known as the Fisher-Kolmogorov equation.)*

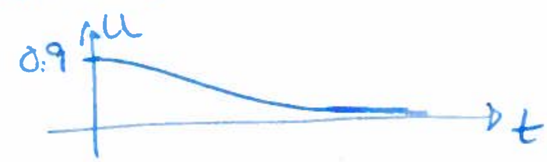
mass = 1
damping = $c > 0$
"spring cst" = $1-u = k$

Note that as long as $0 < u < 1$, $k > 0$ & the "spring" provides a restoring force.

At $t=0$, $u(0) = 0.9$, $k = 0.1 > 0$, so the mass is pulled toward $u = 0$.

Case 1: c large

If c is sufficiently large, the mass won't pick up much speed as u decreases & should just move from 1 to 0:



Case 2: c small

If c is small, the mass will pick up speed & move through $u=0$. It will oscillate back & forth with decaying oscillations: