UBC ID #: ____________ NAME (print): ____________

Signature: ____________

INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

a place of mind UBC

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Apr 21st, 2017 Location: EME 050 Time: 1pm Duration 3 hours.
This exam has 8 questions for a total of 74 points.

SPECIAL INSTRUCTIONS

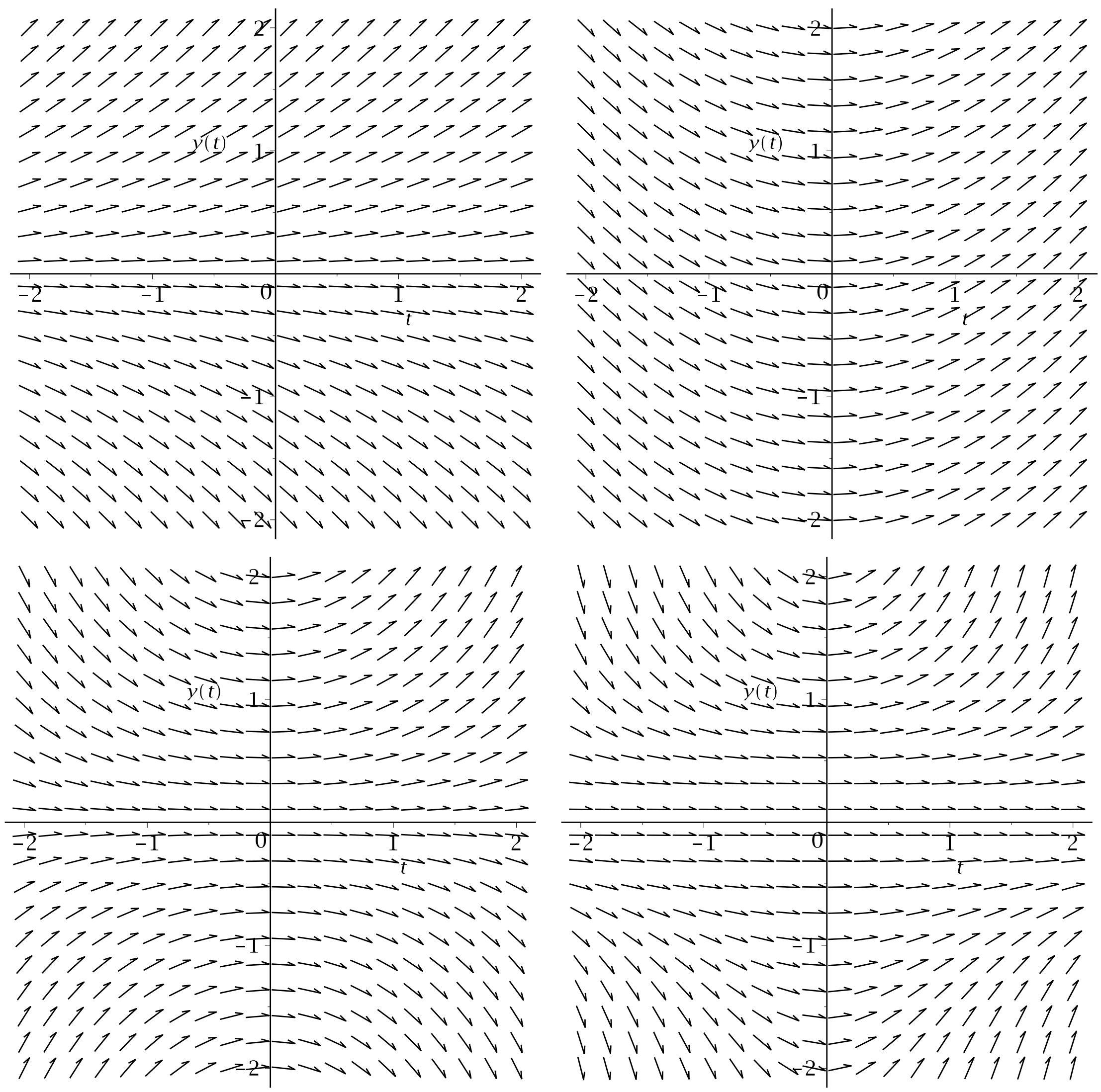
- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	10	4	8	13	9	8	16	74
Score:									

1. Consider the ODE

$$\frac{dy}{dt} = kty^2, \qquad k > 0.$$

$\boxed{2}$ (a) Which of the direction fields corresponds to the given ODE? Circle it. Then explain your choice. Your arguments must be based on the general behaviour of the direction field and ODE in each quadrant, not simply at particular points.



$\boxed{4}$ (b) Solve the ODE. What method are you using?

2. A tank, of volume 600 L, is initially filled with $x(0) = 1000$ g of salt. At time $t = 0$, inflow and outflow valves are opened. The concentration of salt in the inflow is given by

$$(2 + \sin(t) \text{ g/L} .$$

The inflow and outflow rates are both 3 L/min. Let $x(t)$ be the amount of salt in the tank at time t.

$\boxed{2}$ (a) Write down the ODE that describes the change in the amount of salt in the tank at time t.

$\boxed{8}$ (b) Assuming the solution is kept well-stirred, determine the amount of salt in the tank at all times $t > 0$. (Work in **fractions**, not decimals.) *Hint: You may find a useful integral on the formula sheet.*

Workspace for problem #2.

3. Consider the initial value problem

$$\frac{du}{dt} = 2u, \qquad u(0) = 1. \tag{1}$$

1 (a) Write the Backward Euler formula for the given ODE, and solve for u_{n+1}.

3 (b) Using the Backward Euler method, and $h = \Delta\, t = 0.1$, fill in the table below. Show your work!

n	u_n	u_{n+1}
0	1	
1		

4. Consider the ODE

$$t^2y'' - t(t+2)y' + (t+2)y = 0. \tag{2}$$

One solution of (2) is $y_1(t) = t$.

7 (a) Use reduction of order to obtain a second linearly independent solution. (Use the full method shown in class - no short cuts!)

1 (b) Write the general solution.

5. Consider the mass-spring system governed by

$$\frac{d^2y}{dt^2} + y = 5\sin(t), \qquad y(0) = 0, \quad y'(0) = 1.$$

$\boxed{10}$ (a) Determine the equation of motion using the Method of Undetermined Coefficients.

Workspace for problem #5.

$\boxed{2}$ (b) Sketch the solution.

$\boxed{1}$ (c) Name the behaviour.

$\boxed{9}$ 6. Find the general solution of the ODE

$$y'' - 2y' + y = \frac{e^t}{t}, \qquad t > 0.$$

Use fractions, not decimals, in your work and answer. *Use the full method shown in class - no short cuts! Also, make sure that your solution does not contain any redundancies.*

7. Consider the periodic function defined by

$$f(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \end{cases} \quad \text{and } f(t) \text{ has period 2.} \tag{3}$$

$\boxed{2}$ (a) Sketch three periods of the function $f(t)$. Include axis labels.

$\boxed{6}$ (b) Find the Laplace transform of $f(t)$.

8. Consider the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = y'(0) = 0,$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 5, \\ \frac{t-5}{5}, & 5 \le t < 10, \\ 1, & t \ge 10. \end{cases}$$

$\boxed{2}$ (a) The function $g(t)$ is known as "ramp loading". Sketch the function. Include axis labels.

$\boxed{4}$ (b) Express $g(t)$ as a sum of Window and Heaviside functions, then as a sum of only Heaviside functions.

$\boxed{10}$ (c) Solve the IVP using the method of Laplace Transforms.

Workspace for problem #8.

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, ...$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$\delta(t-a)$	$e^{-as}, \quad s > 0$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s-a)$$
$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)H(t-a),$$

where $H(t)$ is the Heaviside function.

TRANSFORM OF A PERIODIC FUNCTION

If f has period T and is piecewise continuous on $[0, T]$, thten the Laplace transforms

$$F(s) = \int_0^\infty e^{-st} f(t)dt \text{ and } F_T(s) = \int_0^T e^{-st} f(t)dt$$

are related by

$$F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$

A POSSIBLY USEFUL INTEGRAL:

$$\int e^{bx} \sin(ax)dx = \frac{1}{a^2 + b^2} e^{bx} \left(b \sin(ax) - a \cos(ax)\right) + C$$