UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 8th, 2017 Time: 11:30am Duration: 35 minutes.
This exam has 5 questions for a total of 20 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

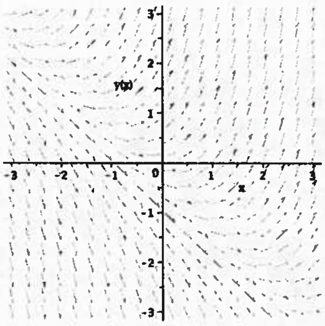
This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Question:	1	2	3	4	5	Total
Points:	4	4	4	6	2	20
Score:						

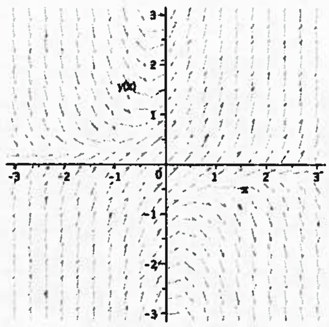
$\boxed{4}$ 1. Consider the two ODEs

$$\boxed{\text{A}}\ \frac{dy}{dx} = 1 - xy, \qquad \boxed{\text{B}}\ \frac{dy}{dx} = x + y.$$

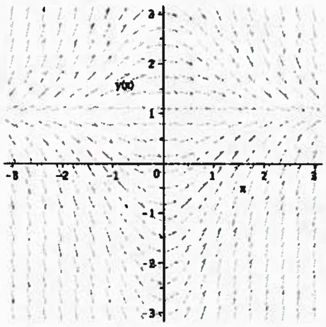
For each ODE, determine the corresponding direction field below, and justify your choice.



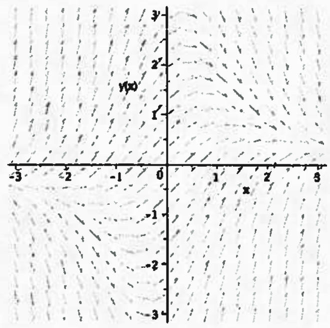
(a) $\boxed{B}$



(b)



(c)



(d) $\boxed{A}$

$\frac{dy}{dx} = 1 - xy = 0$ for $xy = 1$

Only (d) has this property.

$\frac{dy}{dx} = x + y = 0$ for $y = -x$

Only (a) has this property.

$\boxed{4}$ 2. Show that x^4y^3 is an integrating factor for the ODE

$$(3y^2 - 6xy)\,dx + (3xy - 4x^2)\,dy = 0.$$

Check for exactness:

$$\frac{\partial}{\partial y}(3y^2 - 6xy) = 6y - 6x \neq \frac{\partial}{\partial x}(3xy - 4x^2) = 3y - 8x$$

$\therefore$ the ODE is not exact. If we multiply the ODE by x^4y^3, the test for exactness becomes

$$\left.\begin{aligned}\frac{\partial}{\partial y}(3x^4y^5 - 6x^5y^4) &= 15x^4y^4 - 24x^5y^3 \\ \frac{\partial}{\partial x}(3x^5y^4 - 4x^6y^3) &= 15x^4y^4 - 24x^5y^3\end{aligned}\right\}$$ These two expressions are equal!

Thus, the ODE when multiplied by x^4y^3 is exact, & so x^4y^3 is an integrating factor for the ODE.

$\boxed{4}$ 3. Solve the initial value problem

$$\frac{dy}{dx} = 3x^2(1 + y^2), \quad y(0) = 1.$$

Use separation of variables:

$$\frac{dy}{1+y^2} = 3x^2\,dx \iff \int \frac{1}{1+y^2}\,dy = \int 3x^2\,dx \iff$$

$$\iff \arctan(y) = x^3 + C$$

Apply the IC:

$$y(0) = 1 \iff \arctan(1) = C \iff C = \frac{\pi}{4} + n\pi, \; n \in \mathbb{Z} \setminus 0$$

We can choose $C = \frac{\pi}{4}$, & then

$$y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)$$

[6] 4. Find the general solution to the ODE

$$\frac{dr}{d\theta} = -r\tan(\theta) + \sec(\theta).$$

This is a linear ODE in $r(\theta)$. In standard form we have

$$\frac{dr}{d\theta} + r\tan(\theta) = \sec(\theta).$$

The integrating factor is

$$\mu(\theta) = \exp\left[\int \tan(\theta)\,d\theta\right] = \exp\left[-\int \frac{1}{u}\,du\right]$$ where $u = \cos(\theta)$, $du = -\sin(\theta)d$

$$= \exp\left[-\ln|u|\right] = \exp\left[\ln\left|\frac{1}{u}\right|\right]$$

$$= \exp\left[\ln\left|\frac{1}{\cos(\theta)}\right|\right] = \left|\frac{1}{\cos(\theta)}\right|$$

Choose $\mu(\theta) = \frac{1}{\cos(\theta)}$ b/c multiplication by -1 makes no difference. Then the ODE becomes

$$\frac{d}{d\theta}\left[\frac{r}{\cos(\theta)}\right] = \frac{\sec(\theta)}{\cos(\theta)} \iff \frac{r}{\cos(\theta)} = \int \frac{1}{\cos^2(\theta)}\,d\theta = \tan(\theta) + C$$

$$\iff r = \sin(\theta) + C\cos(\theta)$$

Check:
$$\frac{d}{d\theta}\left[\frac{r}{\cos(\theta)}\right] = \frac{1}{\cos(\theta)}\frac{dr}{d\theta} + \frac{\sin(\theta)}{\cos^2(\theta)} \checkmark$$

[2] 5. Write the Backward Euler approximation for the ODE in question 4.

$$r_{n+1} = r_n + h\left[-r_{n+1}\tan(\theta_{n+1}) + \sec(\theta_{n+1})\right]$$