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THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson Course: MATH 225
Date: April 16th, 2018 Time: 9:00am Duration: 3 hours
This exam has 11 questions for a total of 86 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a closed-book, individual exam.

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	9	5	10	6	4	4	10	12	4	8	14	86
Score:												

[5] 1. (a) Find the general solution of

$$x\frac{dy}{dx} + 2y = x^{-3}.$$

[4] (b) Find the general solution of

$$\frac{1}{2}\frac{dy}{dx} = \sqrt{y+1}\cos(x).$$

Express the solution in explicit form.

[5] 2. Show that when Euler's method is used to approximate the solution of the initial value problem

$$y' = 5y, \qquad y(0) = 1,$$

at $x = 1$, then the approximation with stepsize h is $(1 + 5h)^{(1/h)}$.

3. A brine solution flows at a constant rate of 8 L/min into a large tank that initially held 100 L of water in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. The solution entering the tank contains dissolved salt at a concentration of 0.05 kg/L

7 (a) Let $X(t)$ represent the mass of salt in the tank at time t minutes. Find $X(t)$.

3 (b) When will the concentration of salt in the tank reach 0.02 kg/L?

6 4. Solve the initial value problem

$$y'' + 2y' + 17y = 0, \qquad y(0) = 1, \quad y'(0) = -1$$

4 5. Find the general solution of

$$z''' + 2z'' - 4z' - 8z = 0$$

(Hint: $r^3 + 2r^2 - 4r - 8 = (r-2)(r^2 + 4r + 4)$.)

6. For each of the differential equations below, determine the form of the particular solution (do not evaluate the coefficients!). The homogeneous solution is given.

2 (a) $y'' + 9y = 4t^3 \sin(3t)$ ($y_h(t) = c_1 \cos(3t) + c_2 \sin(3t)$)

2 (b) $y'' - 6y' + 9y = 5t^6 e^{3t}$ ($y_h(t) = c_1 e^{3t} + c_2 t e^{3t}$)

10 7. Find the particular solution to the differential equation

$$y'' + y = \sec(t).$$

8. An 8 kg mass is attached to a spring with spring constant 40 N/m, and the system is at rest. At time $t = 0$, an external force $F(t) = \cos(2t)$ N is applied to the system. The damping constant for the system is 3 Ns/m.

8 (a) Translate the word problem into an initial value problem.

4 (b) Is the system at resonance? Explain.

4 9. Find the Laplace transform of the function $h(t) = e^{-t} t \sin(2t)$.

10. Theory

3 (a) Use Theorem 5 (see the Additional Information pages supplied with this exam) to discuss the existence and uniqueness of a solution to the initial value problem

$$(1+t^2)y'' + ty' - y = \tan(t), \qquad y(t_0) = Y_0, \quad y'(t_o) = Y_1,$$

where t_0, Y_0 and Y_1 are real constants.

5 (b) Prove property P4 (see the Additional Information provided with this exam).

14 11. Use the method of Laplace Transforms to solve the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = -1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} t, & 0 < t < 2, \\ 5, & t > 2. \end{cases}$$

Extra space to work on problem 11.

Additional Information

Theorem 5 Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interavl (a, b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, ...$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

P3: $\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\{f\}(s-a)$

P4: $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

P6: $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$

P7: $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a),$$

where $u(t)$ is the unit step function.

TRANSFORM OF A PERIODIC FUNCTION

If f has period T and is piecewise continuous on $[0,T]$, thten the Laplace transforms

$$F(s) = \int_0^\infty e^{-st}f(t)dt \text{ and } F_T(s) = \int_0^T e^{-st}f(t)dt$$

are related by

$$F(s) = \frac{F_T(s)}{1-e^{-sT}}.$$

CONVOLUTION THEOREM

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s),$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

WINDOW FUNCTION

The window function $\Pi_{a,b}$ can be defined in terms of step functions:

$$\Pi_{a,b}(t) = u(t-a) - u(t-b).$$

A POSSIBLY USEFUL INTEGRAL

$$\int u\sin(au)du = -\frac{u}{a}\cos(au) + \frac{1}{a^2}\sin(au) + C$$