UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: April 16th, 2018 Time: 9:00am Duration: 3 hours
This exam has 11 questions for a total of 86 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a closed-book, individual exam.

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	9	5	10	6	4	4	10	5 ~~12~~	4	8	14	86
Score:												

[5] 1. (a) Find the general solution of

$$x\frac{dy}{dx} + 2y = x^{-3}. \iff \frac{dy}{dx} + \frac{2}{x}y = x^{-4} \quad \ldots (1)$$

Integrating factor: $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$

Multiply (1) by x^2:

$$x^2\frac{dy}{dx} + 2xy = x^{-2} \iff \frac{d}{dx}(x^2y) = x^{-2} \iff x^2y = -\frac{1}{x} + C$$

$$\iff \boxed{y = -\frac{1}{x^3} + \frac{C}{x^2}}$$

[4] (b) Find the general solution of

$$\frac{1}{2}\frac{dy}{dx} = \sqrt{y+1}\cos(x). \iff /1/$$

Express the solution in explicit form.

$$/1/ \iff \frac{dy}{\sqrt{y+1}} = 2\cos(x)\,dx \qquad \text{let } u = y+1, \; du = dy$$

$$\iff \int \frac{du}{\sqrt{u}} = \int 2\cos(x)\,dx$$

$$\iff 2\sqrt{u} = 2\sin(x) + \tilde{C} \iff \sqrt{u} = \sin(x) + C$$

$$\iff \sqrt{y+1} = \sin(x) + C \iff \boxed{y = (\sin(x) + C)^2 - 1}$$

[5] 2. Show that when Euler's method is used to approximate the solution of the initial value problem

$$y' = 5y, \qquad y(0) = 1,$$

at $x = 1$, then the approximation with stepsize h is $(1 + 5h)^{(1/h)}$.

Euler's Method:

$$y_{n+1} = y_n + h\,5y_n = (1+5h)\,y_n$$

$\because$ $y(0) = 1$ we have $y_0 = 1$ and

$$y_1 = (1+5h)\,y_0 = (1+5h)$$
$$y_2 = (1+5h)\,y_1 = (1+5h)^2$$
$$y_3 = (1+5h)\,y_2 = (1+5h)^3$$

we observe a pattern:

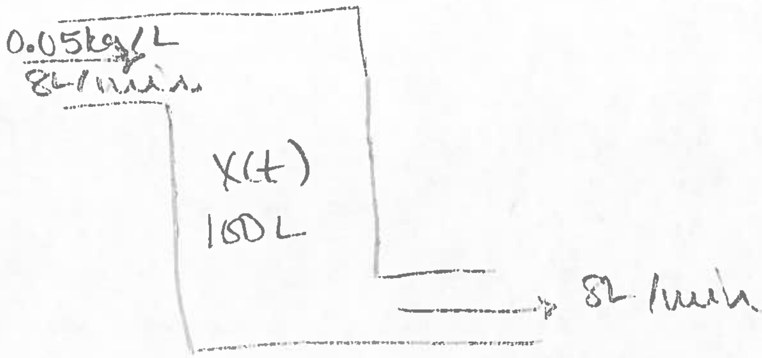
$$y_n = (1+5h)^n \simeq y(n\cdot h)$$

To get to $t=1$, we require $nh = 1$ ~~or~~ $n = \frac{1}{h}$

$$\therefore\ y(1) \simeq y_{1/h} = (1+5h)^{1/h}$$

3. A brine solution flows at a constant rate of 8 L/min into a large tank that initially held 100 L of water in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. The solution entering the tank contains dissolved salt at a concentration of 0.05 kg/L

$\boxed{7}$ (a) Let $X(t)$ represent the mass of salt in the tank at time t minutes. Find $X(t)$.



$X(0) = 0.5\text{kg}$

$$\frac{dX}{dt} = (0.05)8 - \frac{X\,8}{100} \Leftrightarrow \frac{dX}{dt} + \frac{8}{100}X = 0.4 \quad \ldots (2)$$

integrating factor $\mu(t) = e^{\int \frac{8}{100}dt} = e^{\frac{8}{100}t}$

multiply (2) by $\mu(t)$:

$$e^{\frac{8}{100}t}\frac{dX}{dt} + \frac{8}{100}e^{\frac{8}{100}t}X = 0.4e^{\frac{8}{100}t} \Leftrightarrow (I)$$

$$(I) \Leftrightarrow \frac{d}{dt}\left[e^{\frac{8}{100}t}X\right] = 0.4e^{\frac{8}{100}t} \Leftrightarrow e^{\frac{8}{100}t}X = \frac{40e^{\frac{8}{100}t}}{8} + C$$

$$\Leftrightarrow X = 5 + Ce^{-\frac{8}{100}t}$$

$\because X(0) = 0.5\text{kg} \Leftrightarrow 0.5 = 5 + C \Leftrightarrow C = -4.5$

$\therefore X(t) = 5 - 4.5e^{-\frac{8}{100}t}$

[3] (b) When will the concentration of salt in the tank reach 0.02 kg/L?

$$\frac{X(t)}{100} = 0.02 \Leftrightarrow .2 = 5 - 4.5e^{-\frac{8}{100}t}$$

$$\Leftrightarrow 4.5e^{-\frac{8}{100}t} = 5 - 2$$

$$\Leftrightarrow e^{-\frac{8}{100}t} = \frac{3}{4.5}$$

$$\Leftrightarrow -\frac{8}{100}t = \ln\left(\frac{3}{4.5}\right)$$

$$\Leftrightarrow \frac{8}{100}t = \ln\left(\frac{4.5}{3}\right)$$

$$\Leftrightarrow t = \frac{100}{8}\ln(1.5)$$

$$\Leftrightarrow \boxed{t = \frac{25}{2}\ln\left(\frac{3}{2}\right)}$$

[6] 4. Solve the initial value problem

$$y'' + 2y' + 17y = 0, \qquad y(0) = 1, \quad y'(0) = -1$$

characteristic eqn:

$r^2 + 2r + 17 = 0 \iff r = -1 \pm \sqrt{1-17} = -1 \pm 4i$

$\therefore\ y(t) = e^{-t}\left(c_1 \cos(4t) + c_2 \sin(4t)\right)$

apply ICs:

$\begin{cases} y(0) = 1 \\ y'(0) = -1 \end{cases} \iff \begin{cases} c_1 = 1 \\ -c_1 + 4c_2 = -1 \end{cases} \iff \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$

$\therefore\ \boxed{y(t) = e^{-t}\cos(4t)}$

$\boxed{4}$ 5. Find the general solution of

$$z''' + 2z'' - 4z' - 8z = 0$$

(Hint: $r^3 + 2r^2 - 4r - 8 = (r-2)(r^2 + 4r + 4)$.*)*

characteristic eqn:

$$r^3 + 2r^2 - 4r - 8 = 0 \iff (r-2)(r^2+4r+4) = 0$$

$$\iff (r-2)(r+2)^2 = 0$$

$$\therefore\ z(t) = c_1 e^{2t} + c_2 e^{-2t} + c_2 t e^{-2t}$$

6. For each of the differential equations below, determine the form of the particular solution (do not evaluate the coefficients!). The homogeneous solution is given.

$\boxed{2}$ (a) $y'' + 9y = 4t^3 \sin(3t)$ $(y_h(t) = c_1\cos(3t) + c_2\sin(3t))$

or $t(At^3 + Bt^2 + Ct + D)(E\cos(t) + F\sin(t$?

$$y_p(t) = t(A_1 + B_1 t + C_1 t^2 + D_1 t^3)\cos(3t)$$
$$+ t(A_2 + B_2 t + C_2 t^2 + D_2 t^3)\sin(3t)$$

$\boxed{2}$ (b) $y'' - 6y' + 9y = 5t^6 e^{3t}$ $(y_h(t) = c_1 e^{3t} + c_2 t e^{3t})$

$$y_p(t) = t^2(A + Bt + Ct^2 + Dt^3 + Et^4 + Ft^5 + Gt^6)e^{3t}$$

10 7. Find the particular solution to the differential equation

$$y'' + y = \sec(t).$$

hom. prob'm

$y''+y=0 \Rightarrow r^2+1=0$ so $r=\pm i$

$\therefore y_h(t) = c_1\cos(t) + c_2\sin(t)$

part. sol'n

let $y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$

then

$$\begin{cases} v_1'\cos(t) + v_2'\sin(t) = 0 \\ -v_1'\sin(t) + v_2'\cos(t) = \sec(t) \end{cases} \text{ so } \Delta/$$

$$\Delta/ \text{ so } \begin{cases} v_1' = -\sin(t)\sec(t) = -\dfrac{\sin(t)}{\cos(t)} \\ v_2' = \cos(t)\sec(t) = 1 \end{cases} \text{ so } \begin{cases} v_1 = -\displaystyle\int \dfrac{\sin(t)}{\cos(t)}\,dt \\ v_2 = t \end{cases}$$

$$-\int \frac{\sin(t)}{\cos(t)}\,dt = -\int \frac{1}{u}\,du = -\ln|u| = -\ln|\cos(t)|$$

$u = \cos(t)$
$du = \sin(t)\,dt$

$$\therefore \boxed{y_p(t) = -\ln|\cos(t)|\cos(t) + t\sin(t)}$$

8. An 8 kg mass is attached to a spring with spring constant 40 N/m, and the system is at rest. At time $t = 0$, an external force $F(t) = \cos(2t)$ N is applied to the system. The damping constant for the system is 3 Ns/m.

2 $\cancel{\boxed{8}}$ (a) Translate the word problem into an initial value problem.

$$8y'' + 3y' + 40y = \cos(2t), \quad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

3 $\cancel{\boxed{4}}$ (b) Is the system at resonance? Explain.

$$\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{40}{8} - \frac{9}{2 \cdot 64}} = \sqrt{5 - \frac{9}{128}}$$

$$= \sqrt{\frac{640 - 9}{128}} = \sqrt{\frac{531}{128}} \neq 2$$

∴ the forcing frequency (2) is not the same as the resonance frequency ($\sqrt{531/128}$), and so the system is not at resonance.

4 9. Find the Laplace transform of the function $h(t) = e^{-t} t \sin(2t)$.

$\mathcal{L}\{e^{-t} \cdot t \sin(2t)\} = \mathcal{L}\{f\}(s+1)$ where $f(t) = t \sin(2t)$ by P3

$$\mathcal{L}\{f(t)\} = (-1)\frac{d}{ds}\left(\mathcal{L}\{\sin(2t)\}\right) \text{ by P7}$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -2\frac{d}{ds}(s^2+4)^{-1}$$

$$= 2(s^2+4)^{-2}\, 2s$$

$$= \frac{4s}{(s^2+4)^2}$$

$$\therefore \mathcal{L}\{f\}(s+1) = \frac{4(s+1)}{((s+1)^2+4)^2}$$

10. Theory

[3] (a) Use Theorem 5 (see the Additional Information pages supplied with this exam) to discuss the existence and uniqueness of a solution to the initial value problem

$$(1+t^2)y'' + ty' - y = \tan(t), \qquad y(t_0) = Y_0, \quad y'(t_o) = Y_1,$$

where t_0, Y_0 and Y_1 are real constants.

The ODE can be rewritten as $y'' + \underbrace{\frac{t}{1+t^2}}_{p(t)} y' - \underbrace{\frac{1}{1+t^2}}_{q(t)} y = \underbrace{\frac{\tan(t)}{1+t^2}}_{g(t)}$

The functions $p(t)$ + $q(t)$ are defined + continuous $\forall t \in \mathbb{R}$. The domain of $g(t)$ is $\{t \in \mathbb{R} \mid t \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$. Thus, by Thm 5, the solution starting at t_0 exists + is unique as long as $t_0 \neq (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$. The interval of existence is given by $\frac{2n+1}{2}\pi < t_0 < \frac{2n+3}{2}\pi$ for some integer n.

[5] (b) Prove property P4 (see the Additional Information provided with this exam).

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\,dt \qquad \text{let } u = e^{-st} \quad du = -se^{-st}dt, \quad dv = f'(t)dt \quad v = f(t)$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\mathcal{L}\{f\}(s)$$

as required.

14 11. Use the method of Laplace Transforms to solve the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = -1, \quad y'(0) = 0,$$

where

$$g(t) = \begin{cases} t, & 0 < t < 2, \\ 5, & t > 2. \end{cases}$$

$\mathcal{L}\{y''+4y\} = \mathcal{L}\{g(t)\} \Leftrightarrow s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = G(s) \Leftrightarrow$

$\Leftrightarrow (s^2+4)Y(s) = G(s) - s \Leftrightarrow Y(s) = \dfrac{G(s)}{s^2+4} - \dfrac{s}{s^2+4} \quad \cdots (3)$

We can rewrite $g(t)$:

$g(t) = \Pi_{0,2}(t)\,t + u(t-2)5 = t[u(t) - u(t-2)] + 5u(t-2)$

$= t + (5-t)u(t-2) = t - (t-2)u(t-2) + 3u(t-2)$

$\therefore G(s) = \dfrac{1}{s^2} - \dfrac{e^{-2s}}{s^2} + \dfrac{3 \cdot e^{-2s}}{s}$

Plugging $G(s)$ into (3) we obtain

$$Y(s) = \underbrace{\frac{1}{s^2}\frac{1}{s^2+4}}_{T1} - \underbrace{\frac{1}{s^2}\frac{1}{s^2+4}e^{-2s}}_{T2} + \underbrace{\frac{3}{s}\frac{1}{s^2+4}e^{-2s}}_{T3} - \underbrace{\frac{s}{s^2+4}}_{T4}$$

$\mathcal{L}^{-1}\{T1\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2} \cdot \dfrac{1}{2}\dfrac{2}{s^2+4}\right\} = \dfrac{1}{2}(t * \sin(2t)) = \boxed{VI}$

Extra space to work on problem 11.

$$\boxed{VI} = \frac{1}{2}\int_0^t (t-v)\sin(2v)\,dv = \frac{1}{2}\left[t\int_0^t \sin(2v)\,dv - \int_0^t v\sin(2v)\,dv\right]$$

$$= \frac{1}{2}\left[-\frac{t}{2}\cos(2v)\Big|_0^t - \left(-\frac{v}{2}\cos(2v) + \frac{1}{4}\sin(2v)\right)\Big|_0^t\right]$$

$$= \frac{1}{2}\left[-\frac{t}{2}\cos(2t) + \frac{t}{2} - \left(-\frac{t}{2}\cos(2t) + \frac{1}{4}\sin(2t)\right)\right]$$

$$= \frac{t}{4} - \frac{1}{8}\sin(2t)$$

$$\mathcal{L}^{-1}\{T2\} = \mathcal{L}^{-1}\{T1\}(t-2)\,u(t-2) = \left[\frac{t-2}{4} - \frac{1}{8}\sin(2(t-2))\right]u(t-2)$$

$$\mathcal{L}^{-1}\{T3\} = 3\,\mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{s^2+4}\,e^{-2s}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{2}{s^2+4}\,e^{-2s}\right\}$$

$$= \frac{3}{2}\left(1 * \sin(2t)\right)\Big|_{(t-2)} u(t-2) = \frac{3}{2}\left[\int_0^t \sin(2v)\,dv\right]_{(t-2)} u(t-2)$$

$$= \frac{3}{2}\left(-\frac{1}{2}\cos(2v)\right)\Big|_0^{t-2} u(t-2) = -\frac{3}{4}\left[\cos(2(t-2)) - 1\right]u(t-2)$$

$$\mathcal{L}^{-1}\{T4\} = \cos(2t)$$

$$\therefore\ y(t) = \frac{t}{4} - \frac{1}{8}\sin(2t) - \left[\frac{t-2}{4} - \frac{1}{8}\sin(2t-4) + \frac{3}{4}\cos(2t-4) - \frac{3}{4}\right]u(t-2)$$

$$- \cos(2t)$$

Additional Information

Theorem 5 Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interavl (a, b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, ...$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

P3: $\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s-a)$

P4: $\mathcal{L}\left\{f'\right\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

P5: $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$

P7: $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$