UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Jan 31st, 2018 Time: 11:30am Duration: 35 minutes.
This exam has 5 questions for a total of 24 points.

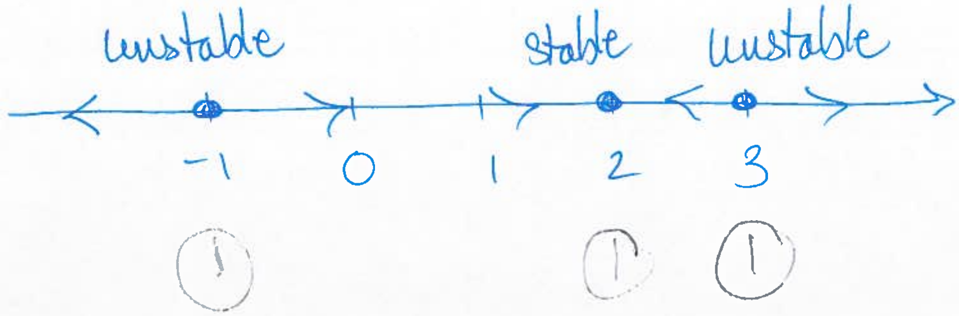
SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Question:	1	2	3	4	5	Total
Points:	3	4	7	3	7	24
Score:						

$\boxed{3}$ 1. Sketch the phase line for the ODE $y' = (y+1)(y-2)(y-3)$ and state the nature of its steady states.



$\boxed{4}$ 2. Find all solutions to the separable ODE

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin(\theta)}{y^2+1}, \quad y(\pi) = 1.$$

(*Possibly useful integral:* $\int \theta \sin(\theta) d\theta = \sin(\theta) - \theta\cos(\theta) + C$)

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin(\theta)}{y^2+1} \Leftrightarrow \frac{y^2+1}{y}\,dy = \theta\sin(\theta)\,d\theta, \quad y \neq 0$$

$$\Leftrightarrow \int\left(y+\frac{1}{y}\right)dy = \int \theta\sin(\theta)\,d\theta$$

$$\Leftrightarrow \frac{y^2}{2} + \ln|y| = \sin(\theta) - \theta\cos(\theta) + C$$

Apply the IC:

$$y(\pi) = 1 \Leftrightarrow \frac{1}{2} + 0 = 0 - \pi + C \Leftrightarrow C = \pi + \frac{1}{2}$$

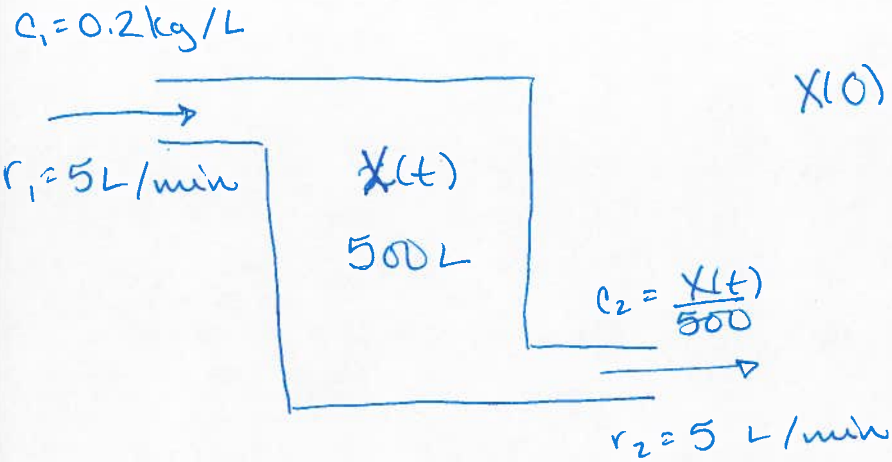
$\therefore$ the solution satisfies

$$\frac{y^2}{2} + \ln|y| = \sin(\theta) - \theta\cos(\theta) + \pi + \frac{1}{2}$$

Note that $y=0$ is not a solution because it does not satisfy the initial condition).

3. Suppose a brine containing salt at a concentration of 0.2 kg/L runs into a tank initially filled with 500 L of water containing 5 kg of salt. The brine enters the tank at a rate of 5 L/min, and the well-stirred mixture flows out at the same rate. Let $X(t)$ be the amount of salt in the tank at time t.

$\boxed{5}$ (a) Make a sketch showing the tank, inflow, and outflow information. Write down the ODE and initial conditions for $X(t)$. Simplify the ODE.



$X(0) = 5\text{ kg}$

$$\frac{dX}{dt} = (5)(0.2) - (5)\left(\frac{X}{500}\right) \iff \frac{dX}{dt} = 1 - \frac{X}{100}$$

with initial condition $X(0) = 5$

$\boxed{2}$ (b) sketch the phase line for the ODE. What is $\lim_{t\to\infty} X(t)$?

$$\lim_{t\to\infty} X(t) = 100\text{ kg}$$

$\boxed{3}$ 4. Find the value of k so that the differential equation below is exact.

$$\underbrace{(y^3 + kxy^4 - 2x)}_{M(x,y)}dx + \underbrace{(3xy^2 + 20x^2y^3)}_{N(x,y)}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Leftrightarrow 3y^2 + 4kxy^3 = 3y^2 + 40xy^3 \Leftrightarrow k = 10$$

$\boxed{7}$ 5. Solve the initial value problem $x\frac{dy}{dx} + 3y + 2x = 3x^2$, $y(1) = 1$.

This is a linear 1^{st} order ODE in y:

$$\frac{dy}{dx} + \frac{3}{x}y + 2 = 3x \Leftrightarrow \frac{dy}{dx} + \frac{3}{x}y = 3x - 2$$

We need an integrating factor

$$\mu = e^{\int \frac{3}{x}dx} = e^{3\ln|x| + \cancel{C}^{0}} = e^{\ln|x^3|}$$

$\because$ we begin @ $x = 1$, choose $x > 0$

Let $\mu(x) = x^3$. Then the ODE becomes

$$x^3\frac{dy}{dx} + 3x^2y = 3x^4 - 2x^3 \Leftrightarrow \text{I}$$

$$\text{I} \Leftrightarrow \frac{d}{dx}\left[x^3y\right] = 3x^4 - 2x^3 \Leftrightarrow x^3y = \frac{3x^5}{5} - \frac{2x^4}{4} + C$$

$$\Leftrightarrow y = \frac{3x^2}{5} - \frac{x}{2} + \frac{C}{x^3}$$

Apply the initial condition $y(1) = 1 \Leftrightarrow 1 = \frac{3}{5} - \frac{1}{2} + C \Leftrightarrow C = \frac{9}{10}$

$$\therefore \boxed{y = \frac{3x^2}{5} - \frac{x}{2} + \frac{9}{10}x^{-3}}$$