UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson Course: MATH 225
Date: Mar 14th, 2018 Time: 11:30am Duration: 35 minutes.
This exam has 4 questions for a total of 32 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Question:	1	2	3	4	Total
Points:	4	5	8	15	32
Score:					

[4] 1. The differential equation $y'' + y = 0$ has the general solution $y(t) = c_1 \cos(t) + c_2 \sin(t)$. Determine the form of the particular solution for the differential equation below (DO NOT SOLVE!):

$$y'' + y = te^{3t} \cos(t) - 4\sin(t)$$

$$y_p(t) = (A + Bt)e^{3t}\cos(t) + (C + Dt)e^{3t}\sin(t)$$
$$+ Et\cos(t) + Ft\sin(t)$$

2. Consider the IVP

$$\frac{dy}{dx} = y(2-x) + x^2, \qquad y(0) = 1.$$

[3] (a) Write out the ODE using the Backward Euler and Forward Euler formulae (do not solve for y_{n+1}).

FE: $\dfrac{y_{n+1} - y_n}{h} = y_n(2 - x_n) + x_n^2$

BE: $\dfrac{y_{n+1} - y_n}{h} = y_{n+1}(2 - x_{n+1}) + x_{n+1}^2$

[2] (b) Your friend chooses to obtain the solution using a different numerical method. After one step of size $h = 0.1$, the magnitude of the local error is ≈ 0.001. What can you say about the method your friend is using? How does it compare to the Backward Euler method?

$0.001 = h^3$ $\therefore$ my friend's method is second order. The BE and FE methods are first order.

3. Consider the mass-spring system with mass 2 kg, damping coefficient 1 kg/s, and spring constant 5/4 N/m. Let $x(t)$ represent the displacement of the mass as a function of time.

$\boxed{2}$ (a) Write down the differential equation for $x(t)$ when the system is subject to the forcing $f(t) = \cos(3t/4)$.

$$2x'' + x' + \frac{5}{4}x = \cos\left(\frac{3t}{4}\right)$$

$\boxed{1}$ (b) Given that the solution to the homogeneous system is

$$x(t) = e^{-\frac{1}{4}t}\left(\cos\left(\frac{3}{4}t\right) + \sin\left(\frac{3}{4}t\right)\right),$$

what is the angular frequency of the homogeneous system?

$$\omega = \frac{3}{4} \ \frac{\text{rad}}{\text{sec}}$$

(c) The general solution of the forced system is

$$x(t) = \overbrace{Ae^{(-b/2m)t}\sin\left(\frac{3}{4}t + \phi\right)}^{\text{term1}} + \overbrace{\frac{8}{\sqrt{37}}\sin\left(\frac{3}{4}t + \theta\right)}^{\text{term2}}, \tag{1}$$

$\boxed{2}$ i. Explain what the two terms in (1) represent.

term 1: transient solution, comes from the hom. soln.
term 2: steady-state solution (long time), comes from the forcing

$\boxed{3}$ ii. What is the frequency gain of the forced system? How does it compare to the amplitude of the forcing itself? Explain.

$M(\gamma) = \frac{8}{\sqrt{37}}$ (from (1)). $\because \frac{8}{\sqrt{37}} > 1$, the steady-state solution amplitude is larger than the amplitude of the forcing. This is because the forcing frequency is the same as the angular frequency of the homogeneous eqn + so the system is close to resonance.

$\boxed{15}$ 4. Solve the initial value problem

$$w'' - 2w' + w = e^s \ln(s), \quad s > 0, \qquad w(1) = e, \quad w'(1) = -e.$$

hom. system

$w'' - 2w' + w = 0$ char. equ: $r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0$

$\therefore$ the homogeneous solution is

$$w_h(s) = c_1 e^s + c_2 s e^s$$

particular sol'n

Let $w_p(s) = v_1(s) e^s + v_2(s) s e^s$. Then v_1 & v_2 must satisfy

$$\begin{cases} v_1' e^s + v_2' s e^s = 0 \\ v_1' e^s + v_2'(1+s)e^s = e^s \ln(s) \end{cases} \Leftrightarrow \begin{cases} v_1' + v_2' s = 0 \\ v_1' + v_2'(1+s) = \ln(s) \end{cases} \Leftrightarrow \updownarrow$$

$$\updownarrow \Leftrightarrow \begin{cases} v_2' = \ln(s) \\ v_1' = -s\ln(s) \end{cases} \Leftrightarrow \begin{cases} v_2 = s\ln(s) - s \\ v_1 = -\int s\ln(s)\,ds \end{cases}$$

To solve for v_1 we use integration by parts

$u = \ln(s) \quad du = \frac{1}{s}ds$

$dv = s\,ds \quad v = \frac{s^2}{2}$

$$\therefore v_1 = -\left[\frac{s^2}{2}\ln(s) - \int \frac{s^2}{2}\frac{1}{s}\,ds\right]$$

$$= -\frac{s^2}{2}\ln(s) + \int \frac{s}{2}\,ds$$

$$= -\frac{s^2}{2}\ln(s) + \frac{s^2}{4}$$

(extra space for problem 4)

(arbitrary csts of integration are set to zero).

$\therefore w_p(s) = \frac{s^2}{2}\left(\frac{1}{2} - \ln(s)\right)e^s + (s\ln(s) - s)se^s$

$w_p'(s) = \left(\frac{s}{2} - \frac{1}{2s}\right)e^s + \frac{s^2}{2}\left(\frac{1}{2} - \ln(s)\right)e^s + (2s\ln(s) + s - 1)e^s + s^2(\ln(s) - 1)e^s$

Now apply the ICs

$$\begin{cases} w(1) = e \\ w'(1) = -e \end{cases} \Leftrightarrow \begin{cases} w_h(1) + w_p(1) = e \\ w_h'(1) + w_p'(1) = -e \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 e + c_2 e + \frac{1}{4}e - e = e \\ c_1 e + 2c_2 e + \frac{1}{4}e - e - e = -e \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 + c_2 = \frac{7}{4} \\ c_1 + 2c_2 = \frac{3}{4} \end{cases} \Leftrightarrow \begin{cases} c_1 = \frac{11}{4} \\ c_2 = -1 \end{cases}$$

$$\therefore w(s) = \frac{11}{4}e^s - se^s + \frac{s^2}{2}\left(\frac{1}{2} - \ln(s)\right)e^s + s^2(\ln(s) - 1)e^s$$

$$= \frac{11}{4}e^s - se^s - \frac{3}{4}s^2e^s + \frac{s^2}{2}\ln(s)e^s$$