UBC ID #: _____ NAME (print): _____

Signature: _____

THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Maria Martignoni Mseya   Course: MATH 225
Date: April 16th, 2019   Time: 1:00pm   Duration: 3 hours
This exam has 10 questions for a total of 75 points.

Rules governing a formal examination

- Each candidate must be prepared to produce a UBCcard for identification.

- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

- No candidate shall be permitted to enter the examination room after the expiration of one half-hour from the scheduled starting time, or to leave during the first half-hour of the examination.

- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action: having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners; speaking or communicating with other candidates; purposely exposing written papers to the view of other candidates or imaging devices.

- Candidates must not destroy or mutilate any examination material, must hand in all examination papers, and must not take any examination material from the examination room.

Special Instructions

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.

- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a closed-book, individual exam.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	9	12	5	5	4	4	8	8	12	75
Score:											

This exam consists of 20 pages including this cover page. Check to ensure that it is complete.

8   1. Consider the equation

$$\frac{dv}{dt} = -a - kv$$

where a and k are positive constant parameters.

(a) Find **all** solutions to the equation above.

(b) Find $\lim_{t \to \infty} v(t)$

(c) Verify the answer with the use of a phase line diagram.

9   2. (a) Find the general solution to

$$\frac{\mathrm{d}^2y}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = 0 \tag{1}$$

(b) Find the solution to Eq. (1) that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$.

(c) Sketch the solution you have obtained in part (b) as a function of time. Your sketch should include the initial conditions and possibly information about the location of maxima and/or minima.

12   3.   (a) Find the general solution to

$$t^2 y'' - 4ty' + 6y = 0 , \quad t > 0$$

(b) Use the results obtained in part (a) to find the general solution to

$$t^2 y'' - 4ty' + 6y = t^3 + 1 , \quad t > 0$$

Note: This is a NON-constant coefficient equation.
Hint: Remember to write the equation in standard form.

Extra space for question 4

5   4. Solve the following IVP by using the method of Laplace transform

$$y''' - y'' + y' - y = 0, \quad y(0) = 1 \ , \quad y'(0) = 1 \ , \quad y''(0) = 1$$

Hint: $a^3 - a^2 + a - 1 = (a-1)(a^2+1)$

5   5.   (a) Write the following IVP

$$y' + \frac{3y}{x} + 2 = 3x \quad , \quad y(1) = 0$$

by using the Forward Euler formula with stepsize h

(b) Express y_1 in terms of h. Simplify your answer and show all your work.

$\boxed{4}$ 6. Verify that when the linear differential equation $[P(x)y - Q(x)]\mathrm{d}x + \mathrm{d}y = 0$ is multiplied by $\mu(x) = e^{\int P(x)\mathrm{d}x}$, the result is exact.

4   7. Use the convolution theorem to find the inverse Laplace transform of

$$F(s) = \frac{3s}{(s^2 + 1)(s + 4)}$$

(Do not solve the integral).

8   8. (a) Find the general solution to the differential equation $y'' + 4y = 0$. Express the general solution in the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

(b) Use the Wronskian to show that $y_1(t)$ and $y_2(t)$ form a fundamental solutions set.

(c) Show that there is no solution that satisfies the conditions $y(0) = 2$ and $y(\pi/2) = 0$.

(d) Explain briefly why your results in part (c) do not contradict the existence and uniqueness theorem (corresponding to Theorem 5, stated at the back of the exam).

8   9. The mixing tank in Figure 1 initially holds 500 L of water (there is initially no salt in the tank). For the first 30 min of operation, valve A is open, adding 5 L/min brine containing a 0.4 Kg/L salt concentration. After 30 min, valve A is closed and valve B is switched in, adding a 0.6 Kg/L concentration at 5 L/min. The exit valve C removes 5 L/min, thereby keeping the volume constant.
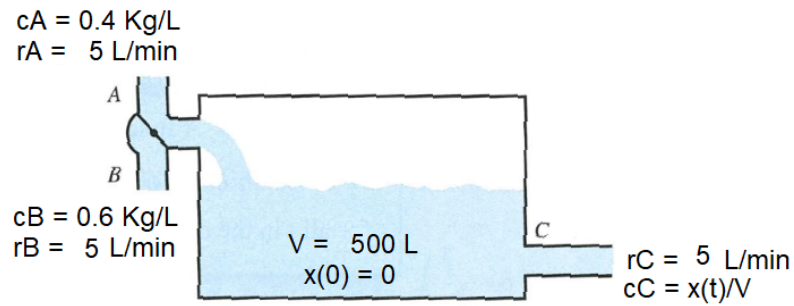
cA = 0.4 Kg/L
rA =   5 L/min

A

B

cB = 0.6 Kg/L
rB =   5 L/min   V =   500 L
                                      x(0) = 0

C

rC = 5 L/min
cC = x(t)/V

Figure 1: Mixing tank

.

(a) Write down the IVP characterising the system (do NOT solve the problem).

(b) Use the method of the Laplace transform to rewrite the IVP found in (a) into the s-space. Solve for X(s).

12 10. (a) Solve the following IVP:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = 5\cos(t), \quad y(0) = 0, \quad y'(0) = 1 \tag{2}$$

Extra space for question 10 (a)

(b) Sketch the solution

(c) Explain briefly what is a possible physical interpretation of the solution you found in part (a). Refer to the forced mass-spring oscillator in your answer.

Empty page

Additional Information

Theorem 5 Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a, b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\dfrac{1}{s}, \quad s > 0$
e^{at}	$\dfrac{1}{s - a}, \quad s > a$
$t^n, \quad n = 1, 2, \dots$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\dfrac{b}{s^2 + b^2}, \quad s > 0$
$\cos(bt)$	$\dfrac{s}{s^2 + b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \dots$	$\dfrac{n!}{(s - a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\dfrac{b}{(s - a)^2 + b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\dfrac{s - a}{(s - a)^2 + b^2}, \quad s > a$
$u(t - a)$	$\dfrac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

P3: $\quad \mathcal{L}\left\{e^{at} f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s - a)$

P4: $\quad \mathcal{L}\left\{f'\right\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

P6: $\quad \mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$

P7: $\quad \mathcal{L}\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a),$$

where $u(t)$ is the unit step function.

TRANSFORM OF A PERIODIC FUNCTION

If f has period T and is piecewise continuous on $[0, T]$, thten the Laplace transforms

$$F(s) = \int_0^\infty e^{-st}f(t)dt \text{ and } F_T(s) = \int_0^T e^{-st}f(t)dt$$

are related by

$$F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$

CONVOLUTION THEOREM

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s),$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

WINDOW FUNCTION

The window function $\Pi_{a,b}$ can be defined in terms of step functions:

$$\Pi_{a,b}(t) = u(t-a) - u(t-b).$$