UBC ID #: ________________ NAME (print): ________________________

Signature: Solutions

THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Maria Martignoni Mseya Course: MATH 225
Date: April 16th, 2019 Time: 1:00pm Duration: 3 hours
This exam has 10 questions for a total of 75 points.

Rules governing a formal examination

- Each candidate must be prepared to produce a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one half-hour from the scheduled starting time, or to leave during the first half-hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action: having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners; speaking or communicating with other candidates; purposely exposing written papers to the view of other candidates or imaging devices.
- Candidates must not destroy or mutilate any examination material, must hand in all examination papers, and must not take any examination material from the examination room.

Special Instructions

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a closed-book, individual exam.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	9	12	5	5	4	4	8	8	12	75
Score:											

8 1. Consider the equation

$$\frac{dv}{dt} = -a - kv$$

where a and k are positive constant parameters.

(a) Find **all** solutions to the equation above.

Constant solution

$$-a - kv = 0$$
$$v = -\frac{a}{k}$$

Assume $-a - kv \neq 0$

Sep. of variables

$$\int \frac{1}{-a-kv}\, dv = \int dt$$

$$-\frac{1}{k} \ln|-a-kv| = t + c$$

$$\ln|-a-kv| = -kt + \tilde{c}t$$

$$-a - kv = \pm e^{\tilde{c}k} e^{-kt}$$

$$A = \pm e^{\tilde{c}k}$$

$$v = A\frac{e^{-kt}}{k} - \frac{a}{k}$$

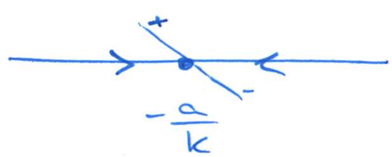
(b) Find $\lim_{t\to\infty} v(t)$

$$\lim_{t\to\infty}\left(A\,\frac{e^{-kt}}{k}-\frac{a}{k}\right)=-\frac{a}{k}$$

(terminal velocity)

(c) Verify the answer with the use of a phase line diagram.

$$\frac{dv}{dt}=0 \iff v=-\frac{a}{k} \quad \text{critical point}$$



stable

Hence, $\lim_{t\to\infty} v(t) = -\frac{a}{k}$

9 2. (a) Find the general solution to

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 9y = 0 \tag{1}$$

$y'' + 2y' + 9y = 0$

$r^2 + 2r + 9 = 0$

$r_{1,2} = \frac{-2 \pm \sqrt{4 - 9 \times 4}}{2} = -1 \pm i2\sqrt{2}$

$y(t) = c_1 e^{-t} \cos(2\sqrt{2}t) + c_2 e^{-t} \sin(2\sqrt{2}t)$

(b) Find the solution to Eq. (1) that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$.

$y(0) = c_1 = 1$

$y'(t) = -e^{-t}\cos(2\sqrt{2}t) - 2\sqrt{2}\, e^{-t}\sin(2\sqrt{2}t)$

$+ c_2\, 2\sqrt{2}\cos(2\sqrt{2}t)\, e^{-t} - c_2 e^{-t}\sin(2\sqrt{2}t)$

$y'(0) = -1 + c_2 2\sqrt{2} = 0$

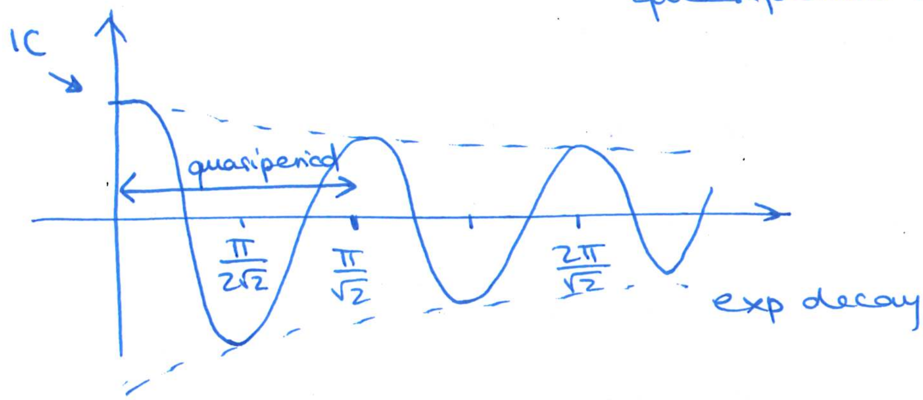
$c_2 = \frac{\sqrt{2}}{4}$ (or $\frac{1}{2\sqrt{2}}$)

$y(t) = e^{-t}\cos(2\sqrt{2}t) + \frac{1}{2\sqrt{2}} e^{-t}\sin(2\sqrt{2}t)$

(c) Sketch the solution you have obtained in part (b) as a function of time. Your sketch should include the initial conditions and possibly information about the location of maxima and/or minima.

quasifrequency $2\sqrt{2}$

quasiperiod : $\frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$



12 3. (a) Find the general solution to

$$t^2y'' - 4ty' + 6y = 0\ , \quad t > 0$$

Cauchy Euler

$$ar^2 + (b-a)r + c = 0$$

$$r^2 + (-4-1)r + 6 = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0 \quad r_1 = 2, \ r_2 = 3$$

$$y(t) = c_1 t^2 + c_2 t^3$$

(b) Use the results obtained in part (a) to find the general solution to

$$t^2y'' - 4ty' + 6y = t^3 + 1\ , \quad t > 0$$

Note: This is a NON-constant coefficient equation.
Hint: Remember to write the equation in standard form.

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = \dfrac{t^3+1}{t^2} \end{cases}$$

$y_1 = t^2, \ y_2 = t^3$

$$\begin{cases} t^3 v_1' + t^2 v_2' = 0 \\ 3t^2 v_1' + 2t v_2' = \dfrac{1}{t^2} + t \end{cases}$$

$v_2' = -t v_1'$

↳ Insert in 2nd eq

Extra space for question 4

$$3t^2 v_1' + 2t^2(-v_1') = \frac{1}{t^2} + t$$

$$t^2 v_1' = \frac{1}{t^2} + t$$

$$v_1' = \frac{1}{t^4} + \frac{1}{t}$$

$$v_1 = \int \frac{1}{t^4}\,dt + \int \frac{1}{t}\,dt = -\frac{1}{3t^3} + \ln(t)$$

$$v_2' = -\frac{t}{t^4} - 1 = -\frac{1}{t^3} - 1$$

$$v_2 = \int -\frac{1}{t^3}\,dt - \int dt = \frac{1}{2t^2} - t$$

$$y_p = v_1 y_1 + v_2 y_2 = \left(-t + \frac{1}{2t^2}\right)t^2 + \left(\ln|t| - \frac{1}{3t^3}\right)t^3$$

$$= -t^3 + \frac{1}{2} + t^3 \ln|t| - \frac{1}{3}$$

$$= t^3(\ln|t| - 1) + \frac{1}{6}$$

$$y_{gen} = \underbrace{c_1 t^2 + c_2 t^3}_{y_{hom}} + \underbrace{t^3(\ln|t| - 1) + \frac{1}{6}}_{y_p}$$

$\boxed{5}$ 4. Solve the following IVP by using the method of Laplace transform

$$y''' - y'' + y' - y = 0, \quad y(0) = 1\,, \quad y'(0) = 1\,, \quad y''(0) = 1$$

Hint: $a^3 - a^2 + a - 1 = (a-1)(a^2+1)$

$$\mathcal{L}\{y'''\} - \mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{y\} = 0$$

$$s^3 Y(s) - s^2 \cancel{- s} \cancel{- 1} - s^2 Y(s) \cancel{+ s} \cancel{+ 1} + sY(s) - 1 - Y(s) = 0$$

$$Y(s)\,(s^3 - s^2 + s - 1) = s^2 + 1$$

$$Y(s) = \frac{s^2+1}{(s^3 - s^2 + s - 1)} = \frac{\cancel{s^2+1}}{(\cancel{s^2+1})(s-1)} = \frac{1}{s-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$$

$\boxed{5}$ 5. (a) Write the following IVP

$$y' + \frac{3y}{x} + 2 = 3x \quad , \quad y(1) = 0$$

by using the Forward Euler formula with stepsize h

$$\frac{y_{n+1} - y_n}{h} = -3\frac{y_n}{x_n} - 2 + 3x_n$$

(b) Express y_1 in terms of h. Simplify your answer and show all your work.

$$y_0 = 0, \quad x_0 = 1$$

$$y_1 = y_0 + h\left(3x_0 - 3\frac{y_0}{x_0} - 2\right)$$

$$= 0 + h(3 - 0 - 2) = h$$

$\boxed{4}$ 6. Verify that when the linear differential equation $[P(x)y - Q(x)]\mathrm{d}x + \mathrm{d}y = 0$ is multiplied by $\mu(x) = e^{\int P(x)\mathrm{d}x}$, the result is exact.

$$\frac{\partial}{\partial x}\left[1\,\mu(x)\right] = \frac{\partial}{\partial y}\left[\mu(x)(P(x)y - Q(x))\right]$$

$$\Leftrightarrow \quad \frac{\partial}{\partial x}\left[e^{\int P(x)dx}\right] = \frac{\partial}{\partial y}\left[e^{\int P(x)dx}P(x)y - e^{\int P(x)dx}Q(x)\right]$$

$$\Leftrightarrow \quad \frac{d}{dx}\left[\int P(x)dx\right]e^{\int P(x)dx} = e^{\int P(x)dx}P(x)$$

$$\Leftrightarrow \quad P(x)\,e^{\int P(x)dx} = P(x)\,e^{\int P(x)dx}$$

✓ exact!

$\boxed{4}$ 7. Use the convolution theorem to find the inverse Laplace transform of

$$F(s) = \frac{3s}{(s^2+1)(s+4)}$$

(Do not solve the integral).

$$F(s) = \frac{3}{s^2+1} \cdot \frac{s}{s+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s+4}\right\}$$

$$= 3\sin(t) * \cos(2t)$$

[8] 8. (a) Find the general solution to the differential equation $y'' + 4y = 0$. Express the general solution in the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$.

Ch. eq. $r^2 + 4 = 0$

$r = \pm 2i$

$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$.

(b) Use the Wronskian to show that $y_1(t)$ and $y_2(t)$ form a fundamental solutions set.

$$\begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(t) \end{vmatrix} = 2(\cos^2(t) + \sin^2(t)) = 2$$

$\neq 0$

$\forall t$

(c) Show that there is no solution that satisfies the conditions $y(0) = 2$ and $y(\pi/2) = 0$.

$y(0) = c_1 = 2$

$y(\frac{\pi}{2}) = c_1(-1) = 0 \Leftrightarrow c_1 = 0$ $\not\!\!\!\!\not\,$

(d) Explain briefly why your results in part (c) do not contradict the existence and uniqueness theorem (corresponding to Theorem 5, stated at the back of the exam).

The existence and uniqueness theorem applies only for initial conditions ($y(t_0) = Y_0$, $y'(t_0) = Y_1$).

In this case we are looking at boundary conditions and a unique solution is therefore not guarantee

$\boxed{8}$ 9. The mixing tank in Figure 1 initially holds 500 L of water (there is initially no salt in the tank). For the first 30 min of operation, valve A is open, adding 5 L/min brine containing a 0.4 Kg/L salt concentration. After 30 min, valve A is closed and valve B is switched in, adding a 0.6 Kg/L concentration at 5 L/min. The exit valve C removes 5 L/min, thereby keeping the volume constant.

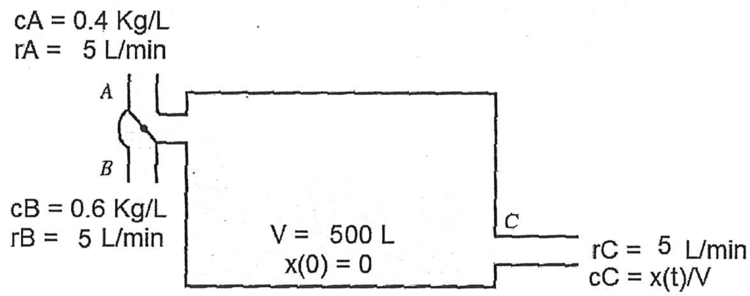


Figure 1: Mixing tank

(a) Write down the IVP characterising the system (do NOT solve the problem).

$$\frac{dx}{dt} = g(t) - \frac{x}{100}$$

where $g(t) = \begin{cases} 2 & \text{for } 0 < t < 30 \\ 3 & \text{for } t > 30 \end{cases}$

or $\frac{d}{dt} x = 2[u(t) - u(t-30)] + 3u(t-30) - \frac{x}{100}$

$$\frac{d}{dt} x = 2u(t) + u(t-30) - \frac{x}{100}$$

(b) Use the method of the Laplace transform to rewrite the IVP found in (a) into the s-space. Solve for X(s).

$$\mathcal{L}\{\dot{x}\} = \mathcal{L}\{2u(t) + u(t-30)\} - \mathcal{L}\{\tfrac{x}{100}\}$$

$$sX(s) - \cancel{x(0)}^{0} = \frac{2}{s} + \frac{e^{-30s}}{s} - \frac{X(s)}{100}$$

$$X(s) = \frac{2 + e^{-30s}}{s(s + \frac{1}{100})}$$

12 10. (a) Solve the following IVP:

$$\frac{d^2y}{dt^2} + y = 5\cos(t), \quad y(0) = 0, \quad y'(0) = 1 \tag{2}$$

$y'' + y = 5\cos(t)$

$r^2 + 1 = 0$

$r = \pm i$

$y_{hom} = c_1 \cos(t) + c_2 \sin(t)$

$y_p = t\,(A\cos(t) + B\sin(t))$

Find A & B

$y_p' = A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t)$

$y_p'' = -2A\sin(t) + 2B\cos(t) - At\cos(t) - Bt\sin(t)$

Insert in the ODE

$-2A\sin(t) + 2B\cos(t) - \cancel{At\cos(t)} - \cancel{Bt\sin(t)} + \cancel{At\cos(t)} + \cancel{Bt\sin(t)} = 5\cos(t)$

$\Longleftrightarrow \quad 2A = 0 \quad \& \quad 2B = 5$

$A = 0$

$B = \frac{5}{2}$

$y_p = \frac{5}{2} t\sin(t)$

Extra space for question 10 (a)

Apply IC

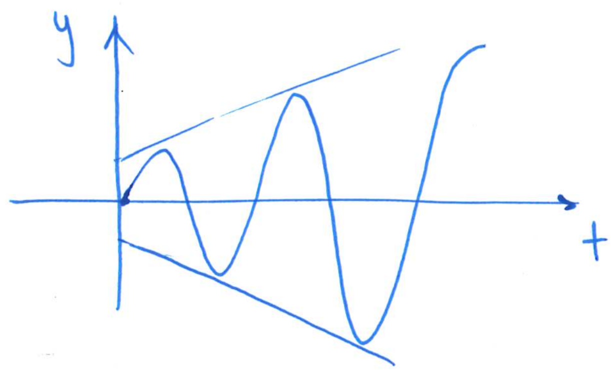
$y(0) = c_1 = 0$

$y'(t) = c_2 \cos(t) + \frac{5}{2}\sin(t) + \frac{5}{2}t\cos(t)$

$y'(0) = c_2 = 1$

$y(t) = \sin(t) + \frac{5}{2}t\sin(t) = \sin(t)\left(1 + \frac{5}{2}t\right)$

(b) Sketch the solution



(c) Explain briefly what is a possible physical interpretation of the solution you found in part (a). Refer to the forced mass-spring oscillator in your answer.

We have a case of resonance.
The frequency of the forcing is the same as the natural frequency of the system.
The amplitude of the oscillations increases with time.

Empty page

Additional Information

Theorem 5 Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on an interval (a,b) that contains the point t_0. Then, for any choice of the initial values Y_0 and Y_1, there exists a unique solution $y(t)$ on the same interval (a,b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \qquad y(t_0) = Y_0, \quad y'(t_0) = Y_1.$$

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

P3: $\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\{f\}(s-a)$

P4: $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

P6: $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$

P7: $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)u(t-a),$$

where $u(t)$ is the unit step function.

TRANSFORM OF A PERIODIC FUNCTION

If f has period T and is piecewise continuous on $[0, T]$, thten the Laplace transforms

$$F(s) = \int_0^{\infty} e^{-st}f(t)dt \text{ and } F_T(s) = \int_0^{T} e^{-st}f(t)dt$$

are related by

$$F(s) = \frac{F_T(s)}{1 - e^{-sT}}.$$

CONVOLUTION THEOREM

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s),$$

or, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

WINDOW FUNCTION

The window function $\Pi_{a,b}$ can be defined in terms of step functions:

$$\Pi_{a,b}(t) = u(t-a) - u(t-b).$$