$\boxed{7}$ 1. Assume that the equation describing the growth population of a size N over time t is given by the solutions of the ODE

$$N' = rN(1 - \frac{N}{K})(N - \mu) \tag{1}$$

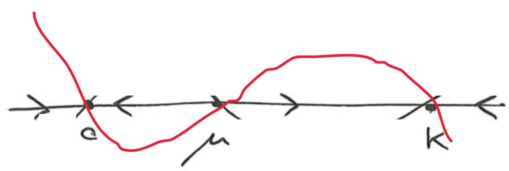
Where μ and K are parameter values, with $\mu \ll K$.

(a) State the independent and the dependent variable of Eq. (1).

N : dependent

t : independent

(b) Sketch the phase line of Eq. (1), and state the nature of the equilibria.



stable unstable stable

(c) What is the behaviour of the solutions for $t \to \infty$? Discuss all possible scenarios.

$N(t) \to K$ for $N_0 > \mu$

$N(t) \to 0$ for $N_0 < \mu$

$(N(t) \to \mu$ for $N_0 = \mu)$

[7] 2. Solve the initial value problem

$$\frac{dy}{dx} = x^3 - 1 + \frac{y}{x}, \quad y(1) = 0$$

The equation is linear.

Standard form: $\frac{dy}{dx} - \frac{y}{x} = x^3 - 1$

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = e^{-\ln(x)} = \frac{1}{x}$$

$$\underbrace{\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y} = x^2 - \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x^2 - \frac{1}{x}$$

$$\frac{y}{x} = \int \left(x^2 - \frac{1}{x}\right)dx$$

$$\frac{y}{x} = \frac{1}{3}x^3 - \ln(x) + C$$

$$y = \frac{1}{3}x^4 - x\ln(x) + Cx$$

Apply IC: $y(1) = \frac{1}{3} + C = 0$

$C = -\frac{1}{3}$

The solution is:

$$y(x) = \frac{1}{3}x^4 - x\ln(x) - \frac{1}{3}x$$

$\boxed{3}$ 3. Match the direction field below with the correct differential equation. SHOW YOUR WORK! (in the absence of any explanation no marks will be given).

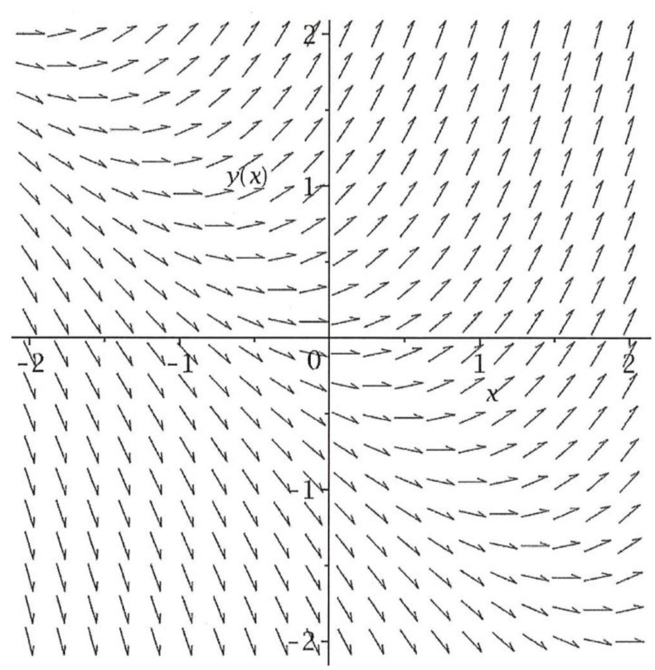
(a) $y' = -x/y$

(b) $y' = xy^2$

(c) $y' = x - y$

(d) $y' = x + y$

(e) $y' = x/y$



Point (1, 1) has a positive slope

~~(a)~~ ~~(e)~~

Point (−1, 1) has zero slope

~~(b)~~ ~~(c)~~

It is (d)

$\boxed{7}$ 4. Consider the equation

$$-\,\mathrm{d}c = k\,c\;\mathrm{d}t, \quad c(0) = c_0 \tag{2}$$

where k is a positive constant.

(a) Solve Eq. (2)

$$\frac{dc}{dt} = -Kc$$

The equation is separable

$$\int \frac{1}{c}\,dc = \int -K\,dt$$

$$\ln|c| = -kt + A$$

$$C = \pm e^{A}\, e^{-kt}$$

Apply IC: $c_{(0)} = \pm e^{A} = c_o$

Solution is $c_{(t)} = c_o\, e^{-kt}$

(b) Assume that the variable c represents a concentration. Find an expression for the half-time, i.e. the time within which the concentration c decreases to 50% of the initial concentration.

$$c_{(t_{1/2})} = \frac{c_o}{2}$$

$$\frac{\cancel{c_o}}{2} = \cancel{c_o}\, e^{-kt_{1/2}}$$

$$e^{-kt_{1/2}} = \frac{1}{2}$$

$$-kt_{1/2} = \ln\left(\frac{1}{2}\right)$$

$$t_{1/2} = \frac{\ln(2)}{K}$$

$\boxed{3}$ 5. Consider the differential equation

$$(2y^2 - \frac{y}{x}\sin x)\,\mathrm{d}x + (xy - \frac{y}{x}\cos y)\,\mathrm{d}y = 0\,. \tag{3}$$

Show that $\mu = \frac{x}{y}$ is an integrating factor of Eq. (3).

We want to show that Eq.(3) multiplied by μ is exact.

$$\underbrace{\left[2xy - \sin(x)\right]}_{M} dx + \underbrace{\left[x^2 - \cos(y)\right]}_{N} dy = 0$$

$$\left.\begin{aligned} \partial_y M &= 2x \\ \partial_x N &= 2x \end{aligned}\right\}$$

The compatibility condition holds,

$\mu = \frac{x}{y}$ is therefore an integrating factor of Eq. (3)