UBC ID #: ____________________ NAME (print): ____

Signature: ____

INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Apr 27th, 2022 Location: EME 050 Time: 7pm Duration: 2 hours 30 minutes.

This exam has 10 questions for a total of 77 points. There is an extra bonus question at the end of the exam.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, you may ask for a clean sheet of paper from the invigilators.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	9	3	3	4	7	12	6	7	16	10	77
Score:											

1. Consider the ODE

$$\frac{dx}{dt} = f(x) = ag(x) + b, \tag{1}$$

where a and b are constants. The function $g(x)$ is plotted in Figure 1.

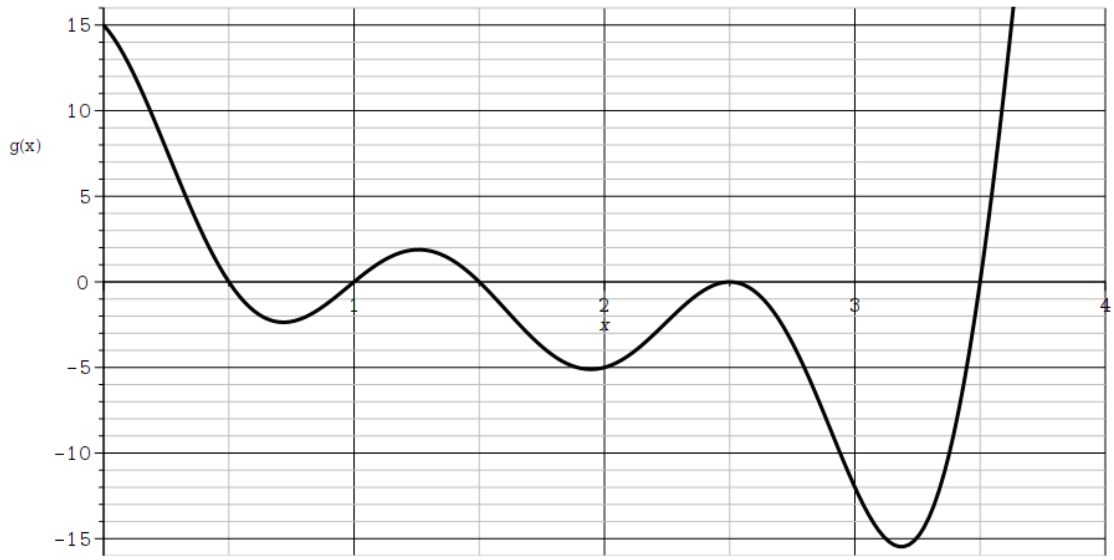
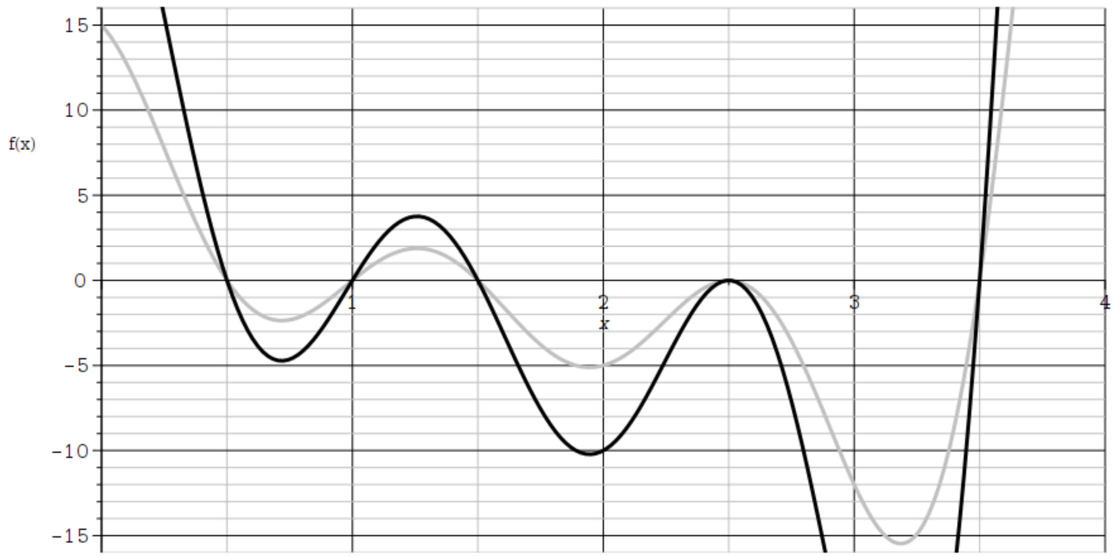


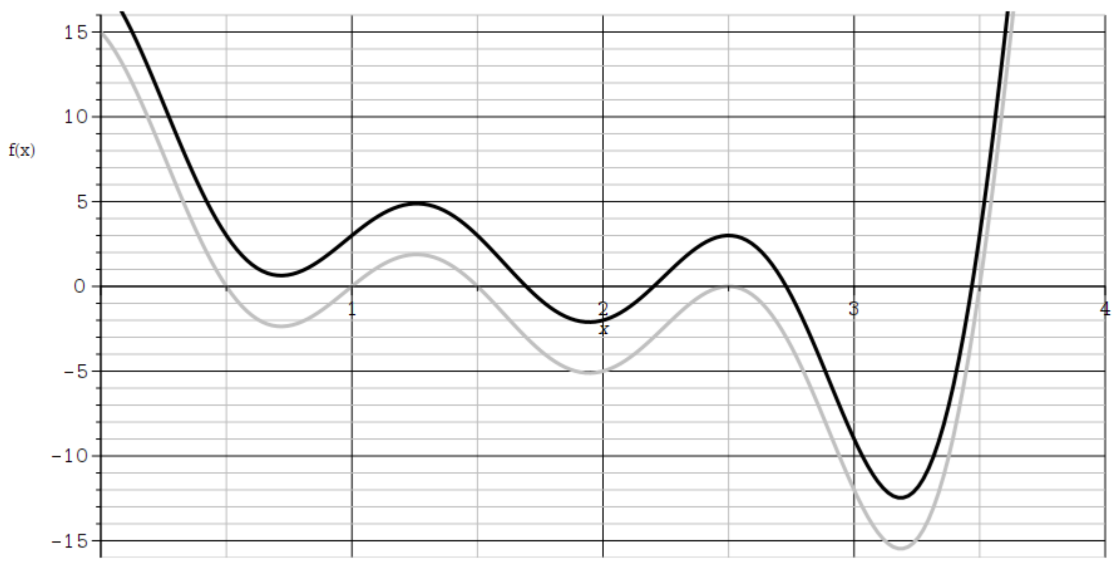
Figure 1: Plot of the function $g(x)$ defined in (1) on the interval $x \in [0, 4]$.

1 (a) We can analyse solutions of (1) using a phase line. Why?

3 (b) Note that the horizontal axis can double as the phase line. On the horizontal axis above, sketch the phase line for equation (1). Label each steady state with its stability.

3 (c) The two plots below show $f_1(x) = 2g(x)$ and $f_2(x) = g(x) + 3$. The plot of $g(x)$ is shown in gray, for reference. Label each plot as appropriate, then sketch the phase line for each case on the horizontal axis.





2 (d) Compare your new phase lines to the original phase line you drew on the previous page. In either case, has a bifurcation occurred? Explain.

[3] 2. Solve the ODE

$$\frac{dy}{dt} = 2y^2 \cos(t)$$

[3] 3. Write the Taylor Series expansion of the function $f(x) = e^{x^2}$ around $x = 0$ up to and including third order.

[4] 4. Consider the IVP $v' = -3 - 2v^2$, $v(0) = 2$. Write the Forward Euler approximation of the ODE, then use this formula to compute $v(0.2)$. Insert your results into the table below (except in the gray cell). Note that you can deduce the value of h from the table.

n	t_n	v_n	$v'(t_n)$
0	0	2	
1	0.1		
2	0.2		

7 5. Consider the IVP

$$(e^{t+y} + 2y)y' + (e^{t+y} + 3t^2) = 0, \quad y(0) = 0.$$

Show that the ODE is exact, and solve the IVP. If possible, write an explicit solution.

12 6. Solve the IVP below using the Method of Undetermined Coefficients. Express your final solution as two phase-shifted sines, and sketch the steady-state solution on the axes provided on the next page.

$$y'' + 4y' + 5y = \cos(t), \qquad y(0) = \frac{9}{8}, \quad y'(0) = \frac{-23}{8}$$

Extra workspace for problem #6.

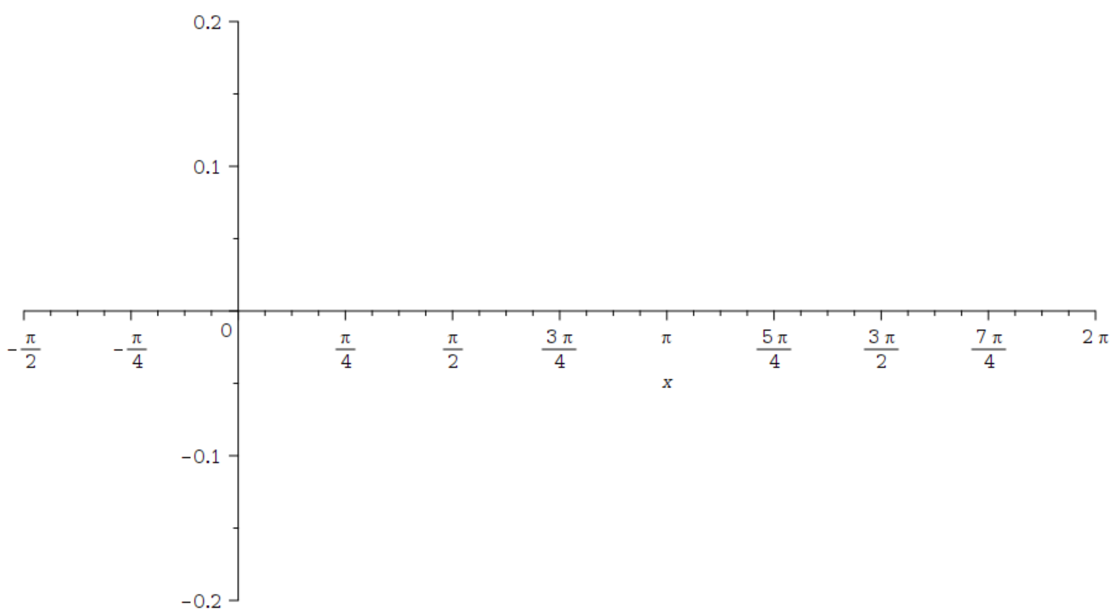
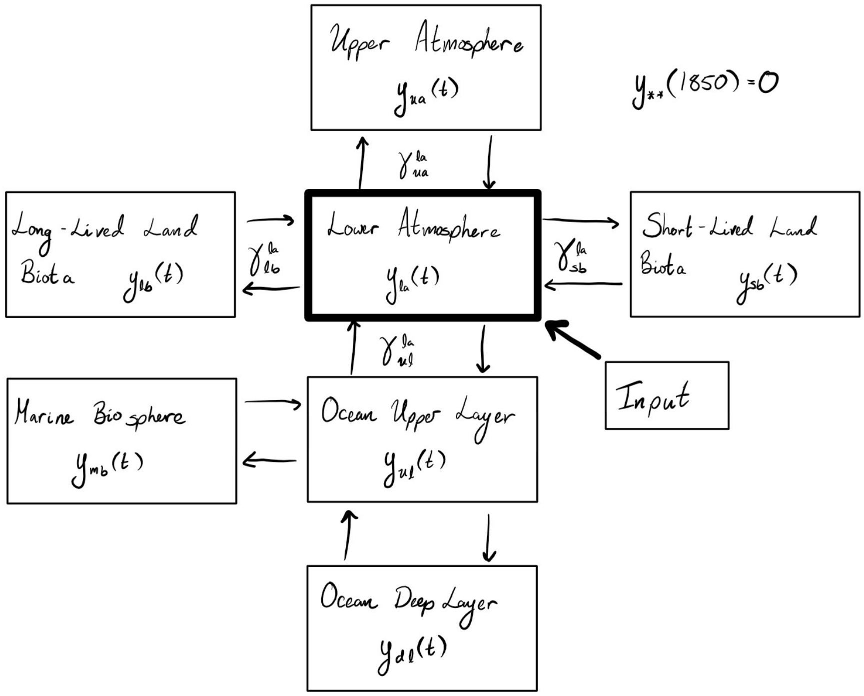


Figure 2: Axes for question #6.

7. Consider the Global CO2 model we studied in class, and shown below.



2 (a) Referring to the compartmental diagram above, write down the ODE for the carbon in the upper atmosphere, $y_{ua}(t)$.

4 (b) Suppose that the carbon in the lower atmosphere is increasing linearly with time at constant rate Q. Then we can write the upper atmospher ODE simply as $y' = Qt - \gamma y$. Solve this ODE.

7 8. Consider the homogeneous ODE $ay'' + by' + cy = 0$, where a, b, and c are real constants. Suppose that ODE has fundamental solution set $\{y_1(t), y_2(t)\}$. Then, a particular solution of $ay'' + by' + cy = f(t)$ is given by $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where

$$v_1'y_1 + v_2'y_2 = 0, \tag{2a}$$
$$v_1'y_1' + v_2'y_2' = \frac{f}{a}. \tag{2b}$$

Derive equations (2).

Hint: Start by plugging $y_p(t)$ into the non-homogeneous ODE.

9. Consider the initial value problem

$$y'' - y = g(t), \quad y(0) = 1, \quad y'(0) = 2, \quad \text{where} \quad g(t) = \begin{cases} 1, & t < 3, \\ t, & t > 3. \end{cases} \tag{3}$$

8 (a) Using the method of Laplace transforms, show that

$$Y(s) = \underbrace{\frac{s+1}{s(s-1)}}_{P1} - \underbrace{\frac{1}{s^2(s+1)}}_{P2} e^{-3s},$$

where the two fractions have been labeled $P1$ and $P2$.

4 (b) Find the partial fraction decompositions of $P1$ and $P2$.

4 (c) Find $y(t)$.

10. Consider the ODE

$$\frac{d^2y}{dt^2} + 9y = 2\cos(3t). \tag{4}$$

6 (a) What is the form of the particular solution of (4)? Sketch the general solution, and name the behaviour.

3 (b) We see an approximation of this behaviour in real systems. Why do we only see an approximation? What can happen in real systems as the behaviour becomes more intense? Explain.

1 (c) Name one application where this behaviour is useful.

11. BONUS question:

2 (a) What does it mean to represent a function by its Taylor Series? Is the representation exact?

1 (b) We used Taylor series multiple times in the course to derive solution methods. Name two of these methods.

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, ...$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$\delta(t-a)$	$e^{-as}, \quad s > 0$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s-a)$$
$$\mathcal{L}\left\{f''\right\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$
$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)H(t-a),$$

where $H(t)$ is the Heaviside function.