UBC ID #: ____________ NAME (print): ____________

Signature: Solutions

INSTRUCTOR: Rebecca Tyson

COURSE: MATH 225

a place of mind UBC

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Date: Apr 27th, 2022 Location: EME 050 Time: 7pm Duration: 2 hours 30 minutes.

This exam has 10 questions for a total of 77 points. There is an extra bonus question at the end of the exam.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, you may ask for a clean sheet of paper from the invigilators.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	9	3	3	4	7	12	6	7	16	10	77
Score:											

1. Consider the ODE

$$\frac{dx}{dt} = f(x) = ag(x) + b, \tag{1}$$

where a and b are constants. The function $g(x)$ is plotted in Figure 1.

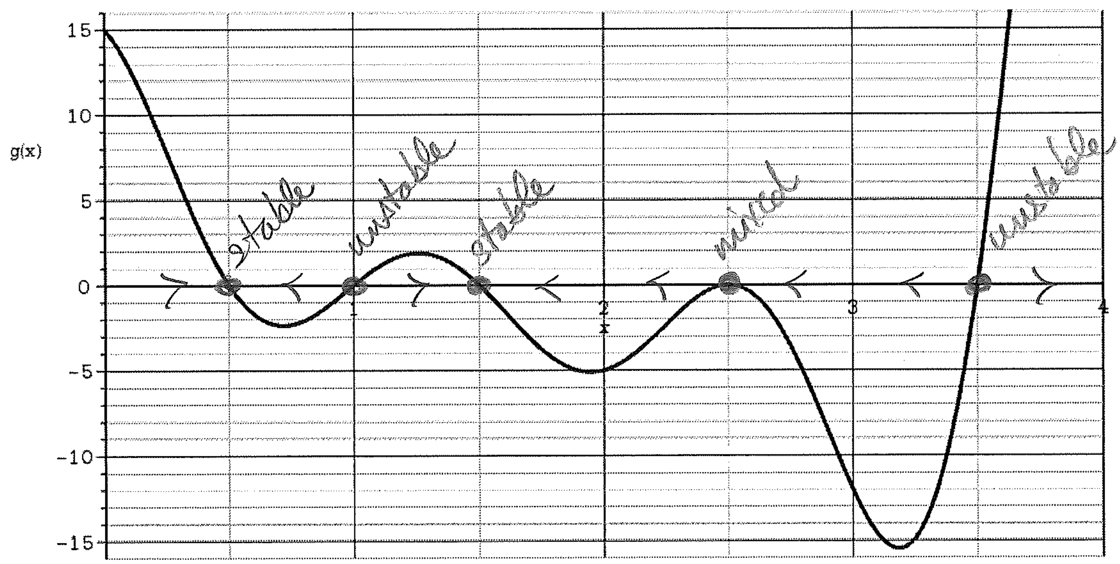


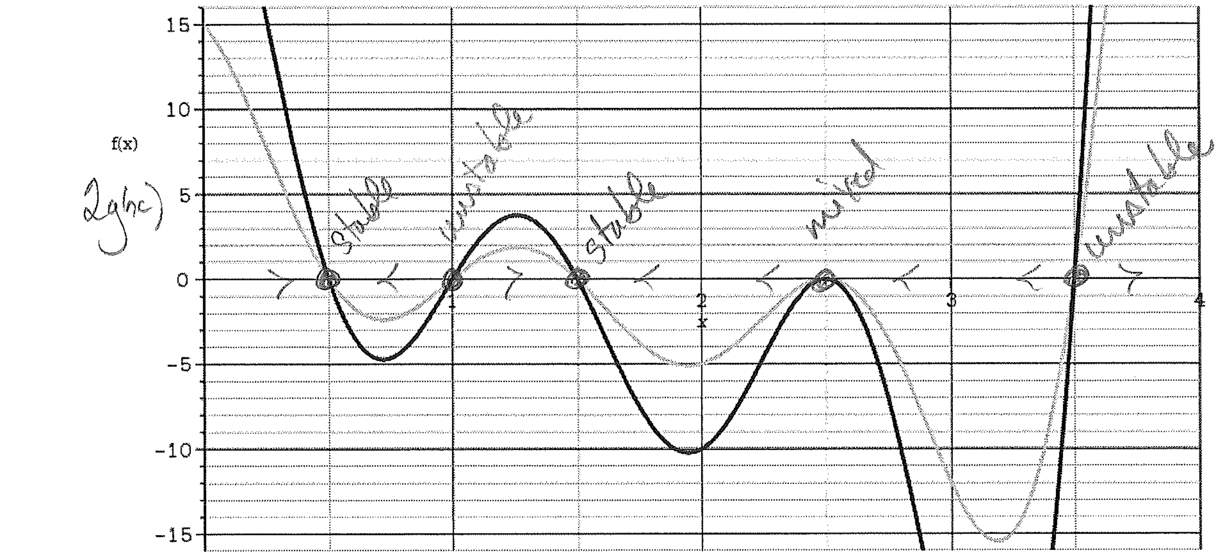
Figure 1: Plot of the function $g(x)$ defined in (1) on the interval $x \in [0, 4]$.

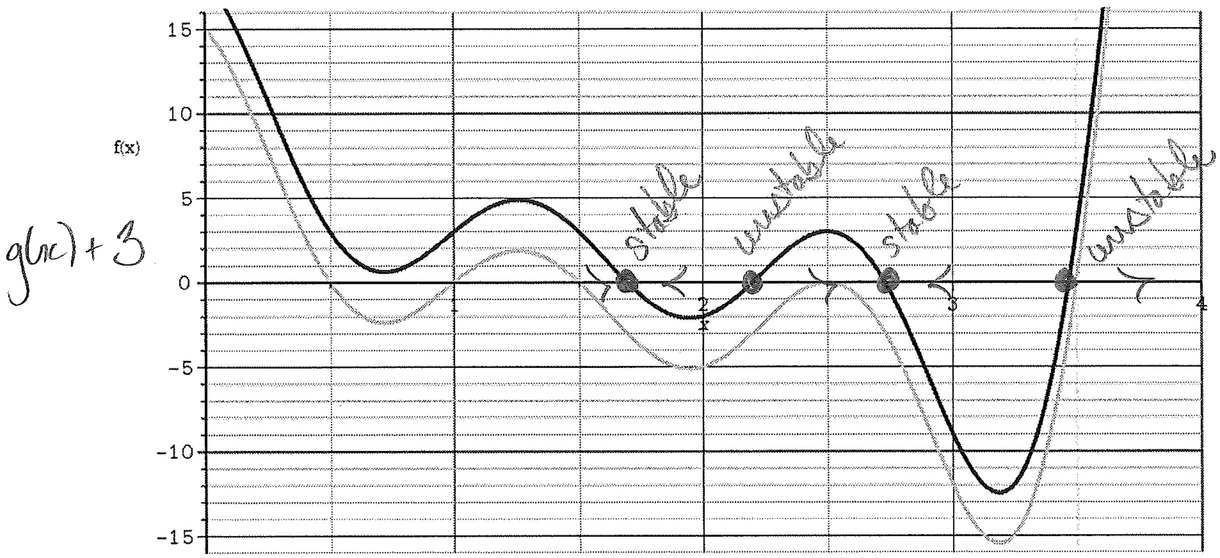
$\boxed{1}$ (a) We can analyse solutions of (1) using a phase line. Why?

Because the ODE is autonomous.

$\boxed{3}$ (b) Note that the horizontal axis can double as the phase line. On the horizontal axis above, sketch the phase line for equation (1). Label each steady state with its stability.

[3] (c) The two plots below show $f_1(x) = 2g(x)$ and $f_2(x) = g(x) + 3$. The plot of $g(x)$ is shown in gray, for reference. Label each plot as appropriate, then sketch the phase line for each case on the horizontal axis.





[2] (d) Compare your new phase lines to the original phase line you drew on the previous page. In either case, has a bifurcation occurred? Explain.

The phase line for 2g(x) is the same as that for g(x).

The phase line for g(x) + 3 is very different though. Here, two bifurcations have occurred: (1) the first two steady states crashed into each other + disappeared, (2) the mixed steady state split into two.

$\boxed{3}$ 2. Solve the ODE

$$\frac{dy}{dt} = 2y^2\cos(t)$$

Separable:

$\frac{dy}{y^2} = 2\cos(t)dt \iff -\frac{1}{y} = 2\sin(t) + C \iff y = \frac{-1}{2\sin(t)+C}$

$\boxed{3}$ 3. Write the Taylor Series expansion of the function $f(x) = e^{x^2}$ around $x = 0$ up to and including third order.

$f'(x) = 2x e^{x^2}$ $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ $f'''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}$
$= 12xe^{x^2} + 8x^3e^{x^2}$

Taylor Series expansion:

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + O(x^4)$

$= 1 + 0\cdot x + \frac{2x^2}{2} + \frac{0\cdot x^3}{3!} + O(x^4)$

$= 1 + x^2 + O(x^4)$

$\boxed{4}$ 4. Consider the IVP $v' = -3 - 2v^2$, $v(0) = 2$. Write the Forward Euler approximation of the ODE, then use this formula to compute $v(0.2)$. Insert your results into the table below (except in the gray cell). Note that you can deduce the value of h from the table.

$h = 0.1$

$v_{n+1} = v_n + h(-3 - 2v_n^2)$

$v'(0) = -3 - 2(2)^2 = -3 - 8 = -11$

$v_1 = 2 + (0.1)(-11) = 0.9$

$v'(0.1) = -3 - 2(0.9)^2 = -4.62$

$v_2 = 1.3 + (0.1)(-4.62) = 0.438$

$\simeq v(0.2)$

n	t_n	v_n	$v'(t_n)$
0	0	2	-11
1	0.1	0.9	$=-4.62$
2	0.2	0.438	

[7] 5. Consider the IVP

$$\underbrace{(e^{t+y} + 2y)}_{M(t,y)}\overset{dy}{y'} + \underbrace{(e^{t+y} + 3t^2)}_{N(t,y)}\overset{dt}{} = 0, \quad y(0) = 0.$$

Show that the ODE is exact, and solve the IVP. If possible, write an explicit solution.

Check:

$$\frac{\partial}{\partial t}(e^{t+y}+2y) = e^{t+y} \qquad \frac{\partial}{\partial y}(e^{t+y}+3t^2) = e^{t+y} \qquad \therefore \text{ the ODE is exact}$$

Solve:

$$\int (e^{t+y}+2y)\,dy = e^{t+y} + y^2 + h(t) = F(t,y)$$

$$\frac{\partial}{\partial t}(e^{t+y} + y^2 + h(t)) = e^{t+y} + 3t^2 \iff h'(t) = 3t^2$$

$$\iff h(t) = t^3$$

$$\therefore F(t,y) = e^{t+y} + y^2 + t^3$$

The solutions are the level curves of F:

$$e^{t+y} + y^2 + t^3 = C.$$

To solve the IVP, we apply the IC:

$$y(0) = 0 \iff 1 = C$$

$$\therefore \boxed{e^{t+y} + y^2 + t^3 = 1}$$ An explicit sol'n is not possible.

12 6. Solve the IVP below using the Method of Undetermined Coefficients. Express your final solution as two phase-shifted sines, and sketch the steady-state solution on the axes provided on the next page.

$$y'' + 4y' + 5y = \cos(t), \qquad y(0) = \frac{9}{8}, \quad y'(0) = \frac{-23}{8}$$

Homogeneous Sol'n:

char eq'n: $r^2 + 4r + 5 = 0 \Leftrightarrow r = -2 \pm \sqrt{4-5} = -2 \pm i$

$\therefore y_h(t) = e^{-2t}\left(c_1 \cos(t) + c_2 \sin(t)\right)$

Particular Sol'n via MOUC:

let $y_p(t) = A\cos(t) + B\sin(t)$

$y_p'(t) = -A\sin(t) + B\cos(t)$

$y_p''(t) = -A\cos(t) - B\sin(t)$

$\therefore y_p'' + 4y_p' + 5y_p = \cos(t) \Leftrightarrow$ //

// $\Leftrightarrow (-A + 4B + 5A)\cos(t) + (-B - 4A + 5B)\sin(t) = \cos(t)$

$\Leftrightarrow \begin{cases} -A + 4B + 5A = 1 \\ -B - 4A + 5B = 0 \end{cases} \Leftrightarrow \begin{cases} 4A + 4B = 1 \\ -4A + 4B = 0 \end{cases}$

$\Leftrightarrow \begin{cases} 8A = 1 \\ A = B \end{cases} \Leftrightarrow \begin{cases} A = 1/8 \\ B = 1/8 \end{cases}$

$\therefore y_p(t) = \frac{1}{8}\left(\cos(t) + \sin(t)\right)$

Extra workspace for problem #6. $y(t) = e^{-2t}\left(c_1\cos(t) + c_2\sin(t)\right) + \frac{1}{8}\left(\cos(t)+\sin(t)\right)$

Apply the ICs:

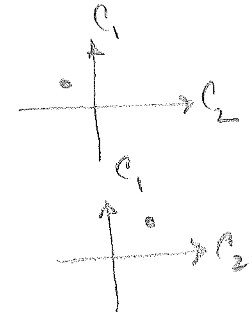
$$\begin{cases} y(0) = \frac{9}{8} \\ y'(0) = -\frac{23}{8} \end{cases} \Leftrightarrow \begin{cases} c_1 + \frac{1}{8} = \frac{9}{8} \\ -2c_1 + c_2 + \frac{1}{8} = -\frac{23}{8} \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = -\frac{24}{8} + 2 = -1 \end{cases}$$

$$\therefore y(t) = e^{-2t}\left(\cos(t) - \sin(t)\right) + \frac{1}{8}\left(\cos(t) + \sin(t)\right)$$

$$= e^{-2t} A_1 \sin(t + \phi_1) + A_2 \sin(t + \phi_2)$$

$A_1 = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ $\quad \tan(\phi_1) = \frac{1}{-1} \therefore \phi_1 = \frac{3\pi}{4}$

$A_2 = \sqrt{\left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2} = \frac{\sqrt{2}}{8}$ $\quad \tan(\phi_2) = \frac{1/8}{1/8} \therefore \phi_2 = \frac{\pi}{4}$



Steady-state sol'n is $y_s(t) = \frac{\sqrt{2}}{8}\sin\left(t + \frac{\pi}{4}\right) \approx 0.18\sin\left(t + \frac{\pi}{4}\right)$

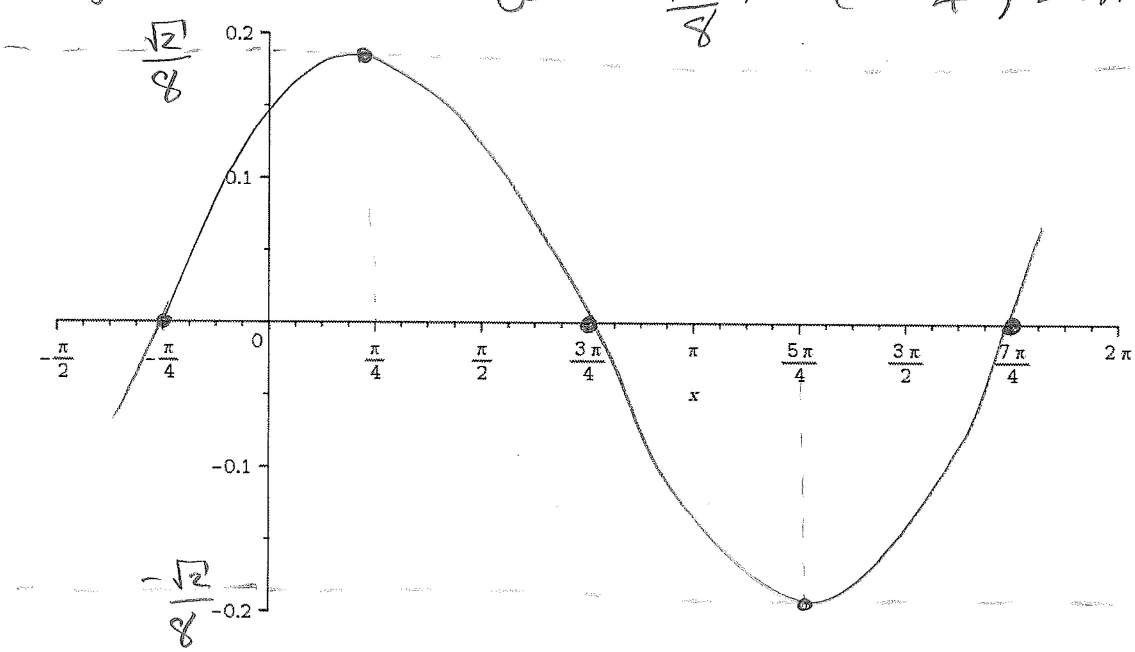
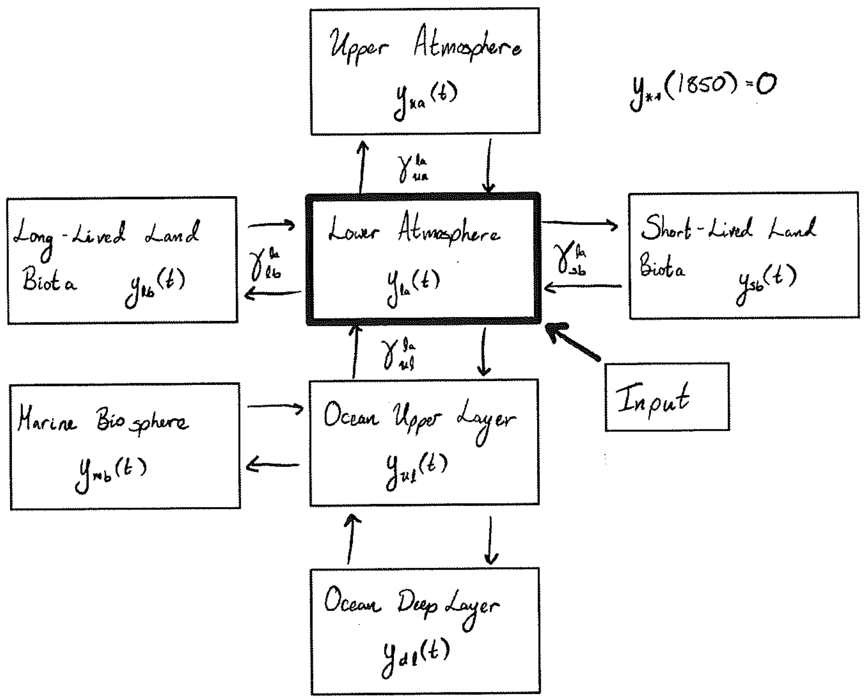


Figure 2: Axes for question #6.

7. Consider the Global CO2 model we studied in class, and shown below.



2 (a) Referring to the compartmental diagram above, write down the ODE for the carbon in the upper atmosphere, $y_{ua}(t)$.

$$\frac{dy_{ua}}{dt} = \gamma^{la}_{ua}\, y_{la} - \gamma^{la}_{ua}\, y_{ua}$$

4 (b) Suppose that the carbon in the lower atmosphere is increasing linearly with time at constant rate Q. Then we can write the upper atmospher ODE simply as $y' = Qt - \gamma y$. Solve this ODE.

$$\frac{dy}{dt} = Qt - \gamma y \iff \frac{dy}{dt} + \gamma y = Qt \iff (1)$$

$$(1) \iff \frac{d}{dt}\left(e^{\gamma t} y\right) = Qte^{\gamma t} \qquad \text{let } u = t,\ dv = e^{\gamma t}dt \qquad du = dt,\ v = \frac{1}{\gamma}e^{\gamma t}$$

$$\iff e^{\gamma t} y = Q\left[\frac{t}{\gamma}e^{\gamma t} - \int \frac{1}{\gamma} e^{\gamma t}\,dt\right] = \frac{Q}{\gamma}\left[t - \frac{1}{\gamma}\right]e^{\gamma t} + C$$

$$\iff y(t) = \frac{Q}{\gamma}\left(t - \frac{1}{\gamma}\right) + Ce^{-\gamma t}$$

7 8. Consider the homogeneous ODE $ay'' + by' + cy = 0$, where a, b, and c are real constants. Suppose that ODE has fundamental solution set $\{y_1(t), y_2(t)\}$. Then, a particular solution of $ay'' + by' + cy = f(t)$ is given by $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where

$$v_1'y_1 + v_2'y_2 = 0, \tag{2a}$$
$$v_1'y_1' + v_2'y_2' = \tfrac{f}{a}. \tag{2b}$$

Derive equations (2).

Hint: Start by plugging $y_p(t)$ into the non-homogeneous ODE.

$y_p(t) = v_1 y_1 + v_2 y_2$

$y_p'(t) = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' = v_1 y_1' + v_2 y_2' + \underbrace{v_1' y_1 + v_2' y_2}_{\text{set to zero}}$

$\therefore\ v_1' y_1 + v_2' y_2 = 0$, which is (2a)

$y_p''(t) = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$

Plugging $y_p, y_p', \&\ y_p''$ into the ODE:

$a y_p'' + b y_p' + c y_p = f(t) \Leftrightarrow a(v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'')$
$+ b(v_1 y_1' + v_2 y_2') + c(v_1 y_1 + v_2 y_2) = f(t) \Leftrightarrow$

$\Leftrightarrow v_1(\cancel{a y_1'' + b y_1' + c y_1})^{0} + v_2(\cancel{a y_2'' + b y_2' + c y_2})^{0}$
$+ a(v_1' y_1' + v_2' y_2') = f(t)$

$\Leftrightarrow v_1' y_1' + v_2' y_2' = \dfrac{f(t)}{a}$, which is (2b)

9. Consider the initial value problem

$$y'' - y = g(t), \quad y(0) = 1, \quad y'(0) = 2, \quad \text{where} \quad g(t) = \begin{cases} 1, & t < 3, \\ t, & t > 3. \end{cases} \tag{3}$$

$\boxed{8}$ (a) Using the method of Laplace transforms, show that

$$Y(s) = \underbrace{\frac{s+1}{s(s-1)}}_{P1} \cancel{- \underbrace{\frac{1}{s^2(s+1)}}_{P2} e^{-3s}}, \quad + \underbrace{\frac{2s+1}{s^2(s^2-1)}}_{P2} e^{-3s}$$

where the two fractions have been labeled $P1$ and $P2$.

Rewrite $g(t)$:

$$g(t) = 1 - H(t-3) + tH(t-3) = 1 + (t-1)H(t-3)$$

$$= 1 + (t-3+2)H(t-3) = 1 + 2H(t-3) + (t-3)H(t-3)$$

Now take the Laplace transform of the ODE:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{g(t)\} \Leftrightarrow s^2 Y - s\,y(0) - y'(0) - Y = \mathcal{L}\{g(t)\} \text{ so,}$$

$$\Leftrightarrow (s^2-1)Y - s - 2 = \frac{1}{s} + 2\frac{e^{-3s}}{s} + \frac{1}{s^2}e^{-3s}$$

$$\Leftrightarrow (s^2-1)Y = s + 2 + \frac{1}{s} + \frac{2e^{-3s}}{s} + \frac{1}{s^2}e^{-3s}$$

$$\Leftrightarrow Y = \frac{1}{(s-1)(s+1)}\left[\left(\frac{s^2+2s+1}{s}\right) + \frac{(2s+1)}{s^2}e^{-3s}\right]$$

$$= \frac{1}{(s-1)(s+1)}\left[\frac{(s+1)^2}{s} + \frac{(2s+1)}{s^2}e^{-3s}\right]$$

$$= \frac{s+1}{(s-1)s} + \frac{2s+1}{s^2(s-1)(s+1)}e^{-3s}$$

$\boxed{4}$ (b) Find the partial fraction decompositions of $P1$ and $P2$.

$$P1 = \frac{A}{s} + \frac{B}{s-1} = \frac{A(s-1)+Bs}{s(s-1)} = \frac{(A+B)s - A}{s(s-1)} = \frac{s+1}{s(s-1)}$$

$$\therefore \begin{cases} A+B=1 \\ -A=1 \end{cases} \iff \begin{cases} B=2 \\ A=-1 \end{cases} \quad \text{so } P1 = -\frac{1}{s} + \frac{2}{s-1}$$

$$P2 = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1} = \frac{As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1)}{s^2(s^2-1)}$$

$$= \frac{(A+C+D)s^3 + (B+C-D)s^2 + (-A)s - B}{s^2(s^2-1)} = \frac{2s+1}{s^2(s^2-1)}$$

$$\therefore \begin{cases} A+C+D=0 \\ B+C-D=0 \\ -A=2 \\ -B=1 \end{cases} \iff \begin{cases} C+D=2 \\ C-D=1 \\ A=-2 \\ B=-1 \end{cases} \iff \begin{cases} A=-2 \\ B=-1 \\ 2C=3 \\ 2D=1 \end{cases} \iff \begin{cases} A=-2 \\ B=-1 \\ C=3/2 \\ D=1/2 \end{cases}$$

$$\text{So } P2 = -\frac{2}{s} - \frac{1}{s^2} + \frac{3}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

$\boxed{4}$ (c) Find $y(t)$.

$$y(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-3s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-3s}\right\}$$
$$+ \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}e^{-3s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}e^{-3s}\right\}$$

$$= -1 + 2e^{t} - 2H(t-3) - (t-3)H(t-3) + \frac{3}{2}e^{(t-3)}H(t-3)$$
$$+ \frac{1}{2}e^{-(t-3)}H(t-3)$$

$$= -1 + 2e^{t} + \left(1 - t + \frac{3}{2}e^{(t-3)} + \frac{1}{2}e^{-(t-3)}\right)H(t-3)$$

10. Consider the ODE

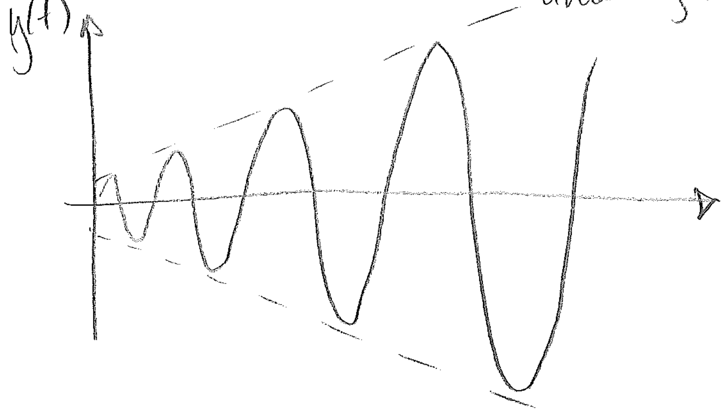
$$\frac{d^2y}{dt^2} + 9y = 2\cos(3t). \tag{4}$$

$\boxed{6}$ (a) What is the form of the particular solution of (4)? Sketch the general solution, and name the behaviour. DO NOT SOLVE FOR THE COEFFICIENTS.

$y_h(t)$: $r^2+9=0 \iff r=\pm 3i$ $\therefore y_h(t)=c_1\cos(3t)+c_2\sin(3t)$

$\therefore y_p(t) = At\cos(3t) + Bt\sin(3t)$ and the general solution is $y(t) = c_1\cos(3t) + c_2\sin(3t) + At\cos(3t) + Bt\sin(3t)$

linearly increasing amplitude



This behaviour is called RESONANCE.

$\boxed{3}$ (b) We see an approximation of this behaviour in real systems. Why do we only see an approximation? What can happen in real systems as the behaviour becomes more intense? Explain.

Real systems always have some damping, so we can only see an approximation of resonance in real systems. With this approximation, however, the amplitude can become so large that it exceeds the system's ability to tolerate oscillations, leading to collapse.

$\boxed{1}$ (c) Name one application where this behaviour is useful.

tuning instruments!

11. BONUS question:

$\boxed{2}$ (a) What does it mean to represent a function by its Taylor Series? Is the representation exact?

Writing a function as a Taylor Series means writing it as a polynomial. The representation is indeed exact, if all terms are included!

$\boxed{1}$ (b) We used Taylor series multiple times in the course to derive solution methods. Name two of these methods.

1) Determining the order of accuracy of numerical methods (Forward Euler, Heun).

2) Determining the meaning of $e^{(\alpha+i\beta)t}$ (i.e. when the characteristic equation has complex roots).

3) Determining the stability of the steady states of a disease model (it depends on the Jacobian).

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$\delta(t-a)$	$e^{-as}, \quad s > 0$
$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

$$\mathcal{L}\left\{e^{at}f(t)\right\}(s) = \mathcal{L}\left\{f\right\}(s-a)$$
$$\mathcal{L}\left\{f''\right\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$
$$\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f\}(s)\right)$$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform of $e^{-at}F(s)$ is given by

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}(t) = f(t-a)H(t-a),$$

where $H(t)$ is the Heaviside function.