UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 9th, 2022 Time: 4:00pm Duration: 35 minutes.
This exam has 4 questions for a total of 28 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

1. Consider the ODE

$$\frac{dy}{dt} = f(y) = y\cos(y) + a, \tag{1}$$

where a is a constant. The function $f(y)$ for the case $a = 0$ is plotted in Figure 1.

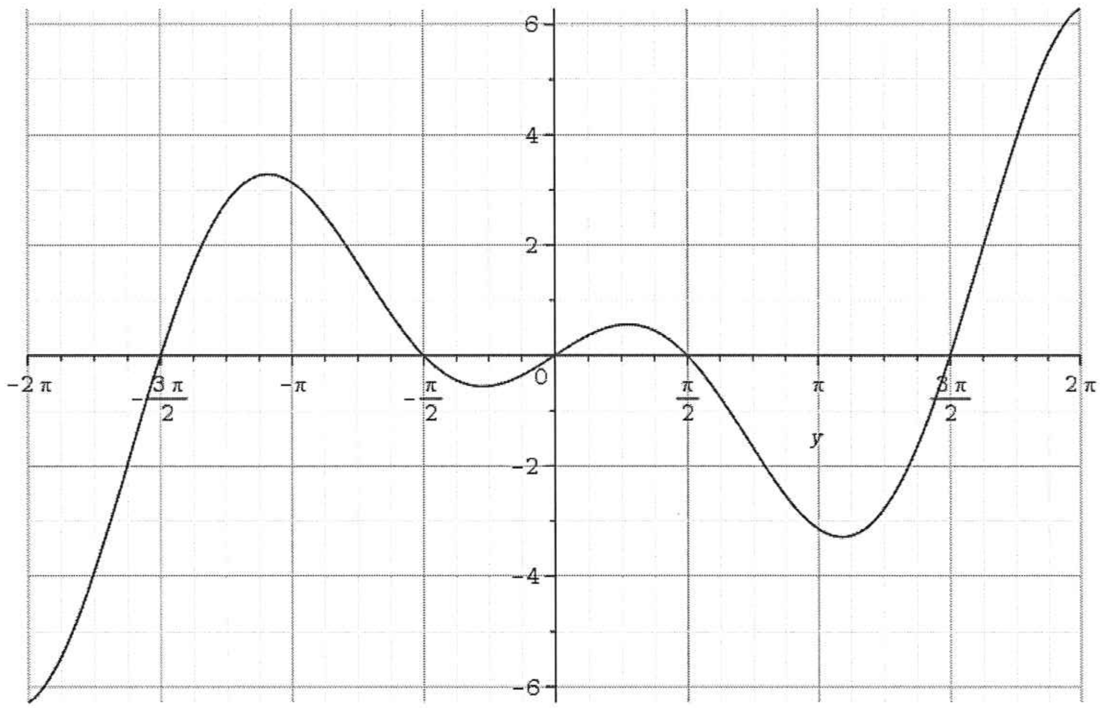
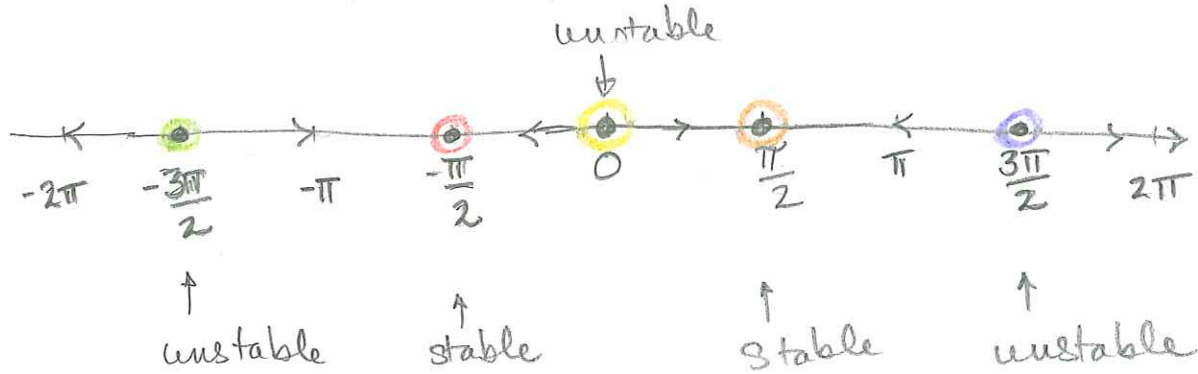


Figure 1: Plot of the function $f(y)$ defined in (1).

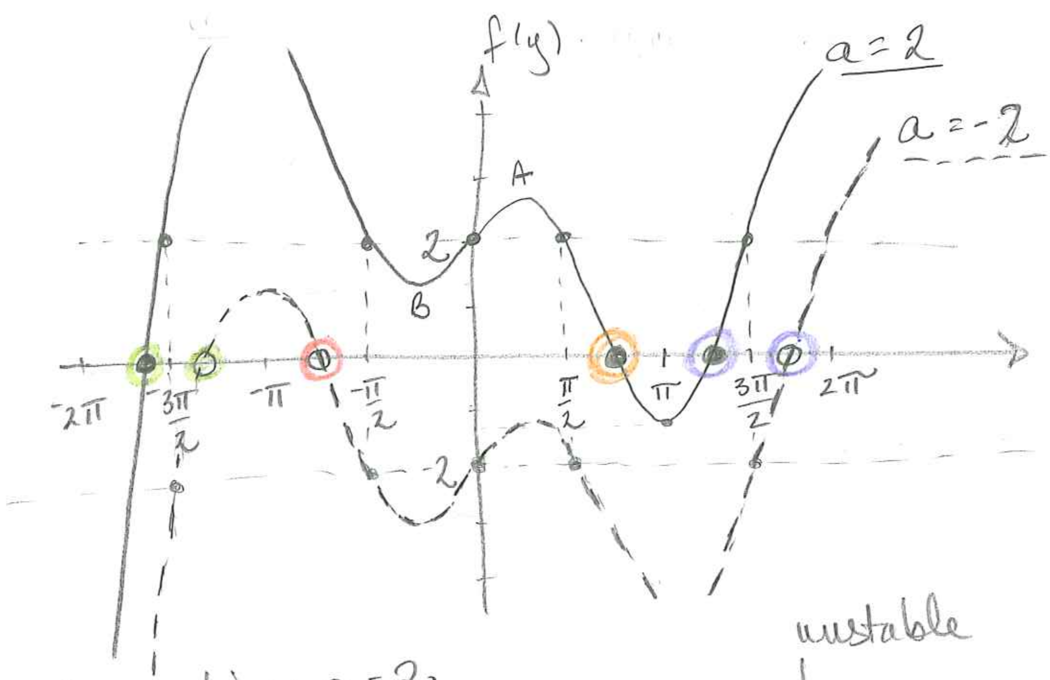
$\boxed{1}$ (a) The ODE (1) is autonomous. Why?

The RHS, $f(y)$, does not depend on t.

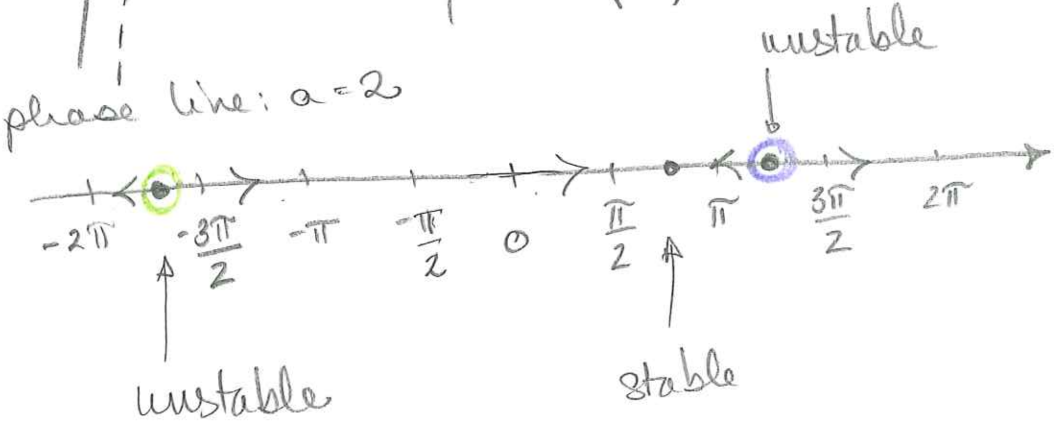
$\boxed{3}$ (b) In the space below, sketch the phase line between $\pm 2\pi$ for the case $a = 0$, and indicate the stability of each steady state.



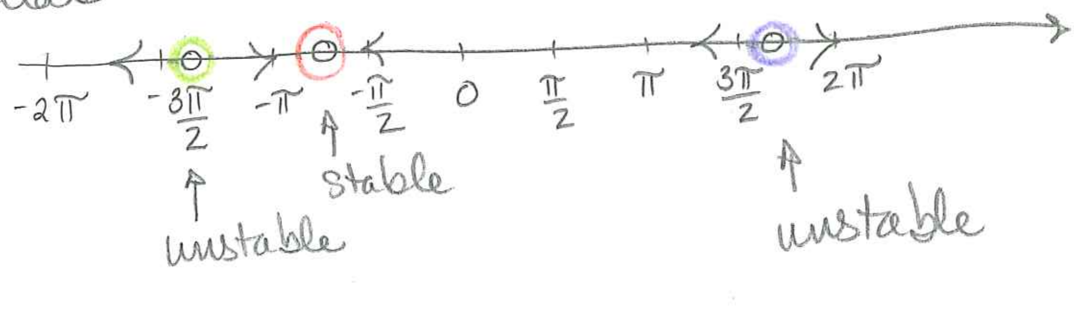
3 (c) How would the stability of the steady states change if $a = 2$ or $a = -2$? Explain.



phase line: a=2



phase line: a=-2



I have colour-coded the steady states for comparison. The green & purple steady states are only shifted L or R as f(y) moves up (a>0) or down (a<0). The other steady states shift or disappear.

$\boxed{7}$ 2. Solve the ODE

$$\underbrace{(\sin(x) + \ln(y))}_{M(x,y)}\, dx + \underbrace{\left(\frac{x}{y} + e^y\right)}_{N(x,y)} dy = 0.$$

Check for exactness:

$\frac{\partial M}{\partial y} = \frac{1}{y}$ $\frac{\partial N}{\partial x} = \frac{1}{y}$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the ODE is exact

Integrate M:

$\int M(x,y)\,dx = \int [\sin(x) + \ln(y)]\,dx = -\cos(x) + x\ln(y) + h(y) = F(x,y)$

Compare with N:

$\frac{\partial F}{\partial y} = \frac{x}{y} + h'(y) = N(y) = \frac{x}{y} + e^y \Rightarrow h'(y) = e^y \Rightarrow h(y) = e^y + C$

We thus have

$F(x,y) = -\cos(x) + x\ln(y) + e^y$

and the solutions are the level curves of F

$$\boxed{-\cos(x) + x\ln(y) + e^y = C}$$

3. Consider the initial value problem

$$\frac{dy}{dt} = te^{-2t} - 2y, \quad y(0) = y_0.$$

5 (a) The ODE is linear. Solve it.

$\frac{dy}{dt} = te^{-2t} - 2y \iff \frac{dy}{dt} + 2y = te^{-2t}$ ← standard form

integrating factor:

$\mu(t) = e^{\int 2dt} = e^{2t}$

multiply by $\mu(t)$:

$e^{2t}\frac{dy}{dt} + 2e^{2t}y = t \iff \frac{d}{dt}\left(e^{2t}y\right) = t$

$\iff e^{2t}y = \frac{t^2}{2} + C$

$\iff y = \frac{t^2}{2}e^{-2t} + Ce^{-2t}$

apply the ICs:

$y(0) = y_0 \iff y_0 = C$

$\therefore \boxed{y(t) = \left(\frac{t^2}{2} + y_0\right)e^{-2t}}$

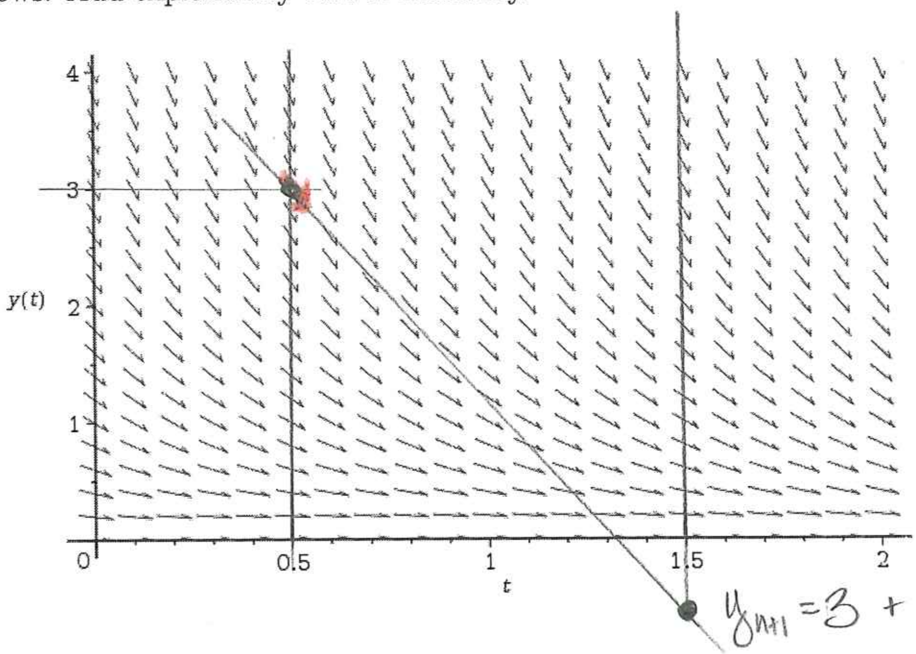
2 (b) How does the value of y_0 affect $\lim_{t\to\infty} y(x)$? Explain.

The value of y_0 has no effect on the final solution, because

$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(\frac{t^2}{2} + y_0\right)e^{-2t} = 0 \ \forall y_0$

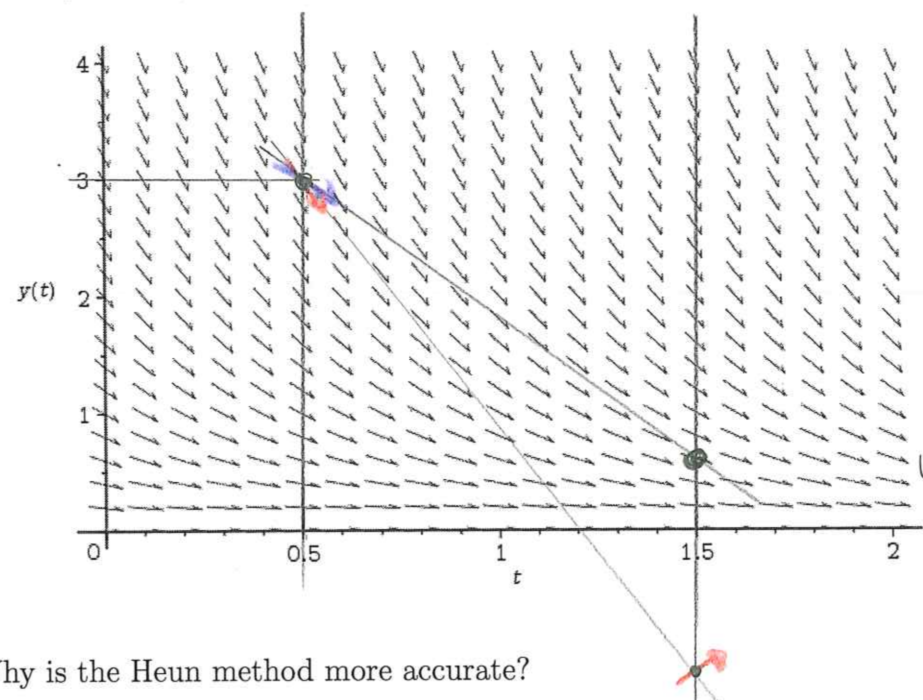
4. Numerical methods.

$\boxed{2}$ (a) On the direction field below, starting at the point $(t, y) = (0.5, 3)$ and using $h = 1$, carefully plot one step of the Forward Euler method, including the relevant slope arrows. Add explanatory text if necessary.



$y_{n+1} = 3 + 1 \cdot \searrow$ (FE)

$\boxed{4}$ (b) On the direction field below, starting at the point $(t, y) = (0.5, 3)$ and using $h = 1$, carefully plot one step of the Heun method, including the relevant slope arrows. Add explanatory text if necessary.



average of the two red arrows is $\simeq \rightarrow$

$y_{n+1} = 3 + \frac{1}{2}(\searrow + \nearrow)$

$= 3 + 1 \cdot \rightarrow$

$\boxed{1}$ (c) Why is the Heun method more accurate?

The Heun method is more accurate because it uses direction field information from two points instead of just one.

Question:	1	2	3	4	Total
Points:	7	7	7	7	28
Score:					