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Signature: Solutions

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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 7th, 2022 Time: 4:00pm Duration: 35 minutes.
This exam has 4 questions for a total of 25 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

1. The direction field for

$$\frac{dy}{dx} = 2x + y \tag{1}$$

is shown in Figure 1. Use it to answer the questions below.

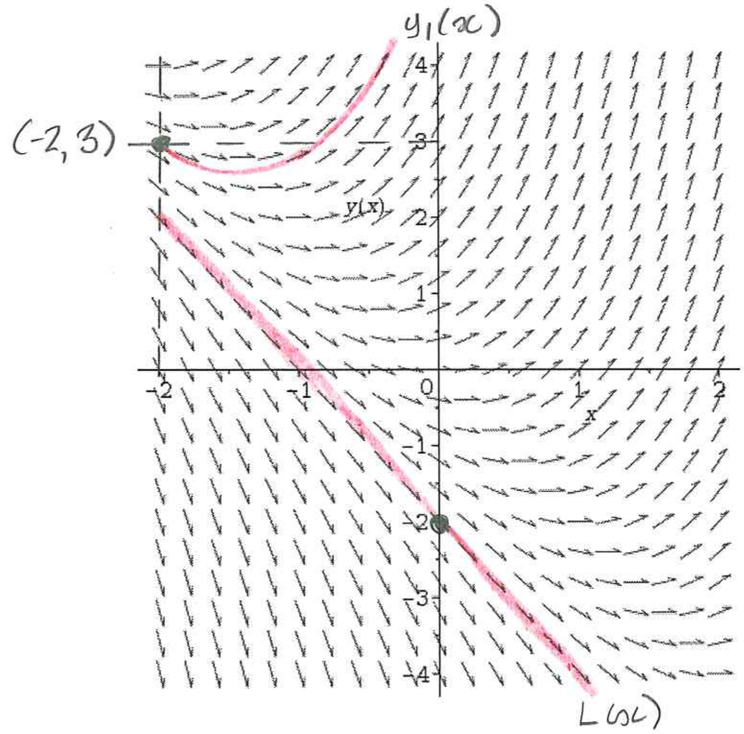

Figure 1: Figure for problem 1.

$\boxed{2}$ (a) Sketch the solution curve that passes through the point (-2,3). Label the curve $y_1(x)$.

$\boxed{1}$ (b) What can you say about $y_1(x)$ as $x \to \infty$?

$$\lim_{x \to \infty} y_1(x) = +\infty$$

$\boxed{2}$ (c) Sketch the solution curve that passes through the point (0,-2) and label it $L(x)$.

$\boxed{1}$ (d) What can you say about the solution $y_1(x)$ as $x \to -\infty$? Be specific.

L(x) appears to be a straight line passing through (−1, 0) and (0, −2). The direction field seems to point along L(x), so

$$\lim_{x \to -\infty} y_1(x) = L(x)$$

4 2. Solve the initial value problem

$$e^x \frac{dy}{dx} = y^2, \quad y(0) = 1,$$

using separation of variables. Show all your work.

$$e^x \frac{dy}{dx} = y^2 \iff \frac{dy}{y^2} = e^{-x}\,dx \iff -\frac{1}{y} = -e^{-x} + C$$

apply the ICs;

$$y(0) = 1 \iff -\frac{1}{1} = -e^{-0} + C \iff -1 = -1 + C \iff C = 0$$

$$\therefore -\frac{1}{y} = -e^{-x} \iff \frac{1}{y} = e^{-x} \iff \boxed{y = e^{x}}$$

3. Consider the ODE

$$\underbrace{(3y^2 - t^2)}_{M}dy \underbrace{- 2ty}_{N}dt = 0. \qquad (2)$$

2 (a) Show that (2) is exact.

$$\frac{\partial}{\partial t} M(t,y) = \frac{\partial}{\partial t}(3y^2 - t^2) = -2t$$

$$\frac{\partial N}{\partial y} N(t,y) = \frac{\partial}{\partial y}(-2ty) = -2t$$

$$\therefore \frac{\partial M}{\partial t} = \frac{\partial N}{\partial y} \text{ the ODE is exact.}$$

7 (b) Solve the ODE (2). You should obtain an implicit solution for $y(t)$.

Integrate wrty:

$$\int M(t,y)\,dy = y^3 - t^2 y + h(t) = F(t,y)$$

Differentiate wrt t & compare to $N(t,y)$:

$$\frac{\partial}{\partial t} F(t,y) = \frac{\partial}{\partial t}\left(y^3 - t^2 y + h(t)\right) = -2ty + h'(t) = -2ty$$

$$\therefore h'(t) = 0 \iff h(t) = C$$

Form F: We thus take

$$F(t,y) = y^3 - t^2 y$$

and observe that the solutions of (2) are the level curves of F:

$$\boxed{y^3 - t^2 y = C}$$

4. . Consider the ODE

$$\frac{dN}{dt} = N(1-N).$$

2 (a) Write the Forward Euler (FE) and Backward Euler (BE) approximations (in the BE case, do not solve for N_{t+1}).

FE: $N_{t+1} = N_t + h \cdot N_t(1 - N_t)$

BE: $N_{t+1} = N_t + h \cdot N_{t+1}(1 - N_{t+1})$

$\boxed{2}$ (b) The direction field for the ODE and the true solution when $N(0) = 0.1$ is shown in Figure 2. Show two steps of the FE and BE methods on the direction field using $h = 2$.

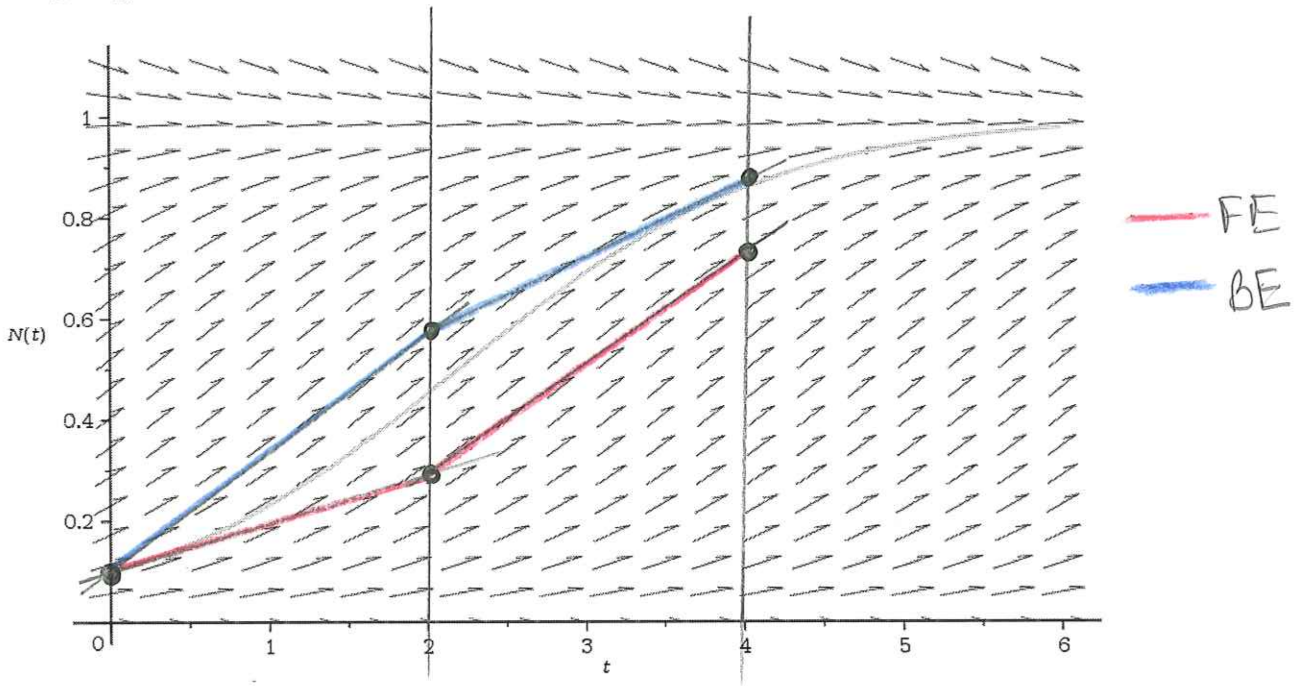

Figure 2: Figure for problem 4.

$\boxed{2}$ (c) Why does the FE method initially underestimate the true solution? Explain.

The FE method only uses the slope at the beginning of each timestep. Over the first timestep, the direction field is increasing, concave up, in the vicinity of the starting point. So the slope at the beginning of the timestep is less than the slope at any later point in the timestep. Since the FE method only uses the initial slope, its estimate of the solution is an underestimate.