UBC ID #: ______________ NAME (print): ______________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Mar 23rd, 2022 Time: 4:00pm Duration: 35 minutes.
This exam has 5 questions for a total of 48 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

1. The homogeneous ODE

$$y'' + 5y' + 6y = 0,$$

has general solution

$$y(t) = c_1 e^{-3t} + c_2 e^{-2t}.$$

$\boxed{2}$ (a) Write the FORM of the particular solution for the ODE $y'' + 5y' + 6y = 3t^2 + 2e^t$.

$$y_p(t) = A + Bt + Ct^2 + De^t$$

$\boxed{2}$ (b) Write the FORM of the particular solution for the ODE $y''+5y'+6y = 5e^{-3t}+2te^{-2t}$.

$$y_p(t) = Ate^{-3t} + Bt^2e^{-2t}$$

$\boxed{4}$ 2. What are the resonance frequency and maximum possible frequency gain for the mass-spring system with mass 4, spring constant 1, and damping coefficient 2 (assume all numbers are given in a consistent system of units)? Your answer should be exact.

γ_r = resonance frequency

$$= \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{1}{4} - \frac{4}{2\cdot 16}}$$

$$= \sqrt{\frac{1}{4} - \frac{1}{8}} = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}}$$

$$M(\gamma_r) = \frac{1/b}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4} - \frac{4}{4\cdot 16}}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4} - \frac{1}{16}}}$$

$$= \frac{1}{2} \frac{1}{\sqrt{\frac{3}{16}}} = \frac{4}{2} \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

3. Consider the differential equation $\frac{1}{2}y'' + 2y = \tan(2t)$, which has fundamental solution set $\{\cos(2t), \sin(2t)\}$.

$\boxed{8}$ (a) Find a particular solution. *This integral may be helpful:* $\int(\tan(2t)\sin(2t)dt) = \frac{1}{2}(-\sin(2t) + \ln(\sec(2t) + \tan(2t)))$

Try

$$y_p(t) = v_1(t)\cos(2t) + v_2(t)\sin(2t)$$

So

$$\begin{cases} v_1'(t)\cos(2t) + v_2'(t)\sin(2t) = 0 \\ -2v_1'(t)\sin(2t) + 2v_2'(t)\cos(2t) = \dfrac{\tan(2t)}{1/2} \end{cases}$$

$$\iff \begin{cases} 2v_2'(\cos^2(2t) + \sin^2(2t)) = 2\tan(2t)\cos(2t) \\ 2v_1'(\cos^2(2t) + \sin^2(2t)) = -2\tan(2t)\sin(2t) \end{cases}$$

$$\iff \begin{cases} v_2' = \sin(2t) \\ v_1' = -\tan(2t)\sin(2t) \end{cases}$$

$$\iff \begin{cases} v_2 = -\frac{1}{2}\cos(2t) \\ v_1 = \frac{1}{2}\left(\sin(2t) - \ln(\sec(2t) + \tan(2t))\right) \end{cases}$$

$$\therefore y_p(t) = \cancel{\tfrac{1}{2}\sin(2t)\cos(2t)} - \tfrac{1}{2}\ln[\sec(2t) + \tan(2t)]\cos(2t) - \cancel{\tfrac{1}{2}\cos(2t)\sin(2t)}$$

$$= -\tfrac{1}{2}\ln[\sec(2t) + \tan(2t)]\cos(2t)$$

$\boxed{2}$ (b) Write the general solution.

$$y(t) = c_1\cos(2t) + c_2\sin(2t) - \tfrac{1}{2}\ln[\sec(2t) + \tan(2t)]\cos(2t)$$

4. A 3 kg mass is attached to a spring with stiffness $k = 75$ N/m. The mass is displaced 1 m to the right and given a velocity of 5 m/sec to the left. The damping force is negligible.

$\boxed{6}$ (a) Find the equation of motion of the mass.

$$3y'' + 75y = 0 \Leftrightarrow y'' + 25y = 0 \Rightarrow r^2 + 25 = 0 \Leftrightarrow r = \pm 5i$$

$$\therefore y(t) = c_1 \cos(5t) + c_2 \sin(5t)$$

$$\begin{cases} y(0) = 1 \\ y'(0) = -5 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases} \quad \therefore y(t) = \cos(5t) - \sin(5t)$$

$\boxed{7}$ (b) Write the solution as a single phase-shifted sine, and sketch it. Include labels for the amplitude, phase shift, and period. The phase shift should be exact and in radians.

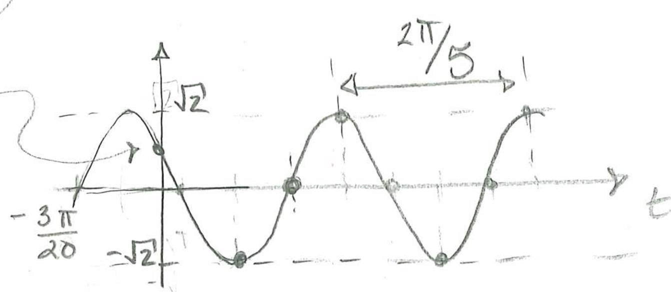
$$A = \sqrt{1^2 + (-1)^2} = \sqrt{2} \qquad \tan(\phi) = \frac{1}{-1}$$

c_1, c_2

$y(0) = 1$
$y'(0) = -5$

$$\therefore \phi = \frac{3\pi}{4} \quad \text{and} \quad y(t) = \sqrt{2} \sin\left(5t + \frac{3\pi}{4}\right)$$

$$= \sqrt{2} \sin\left(5\left(t + \frac{3\pi}{20}\right)\right)$$



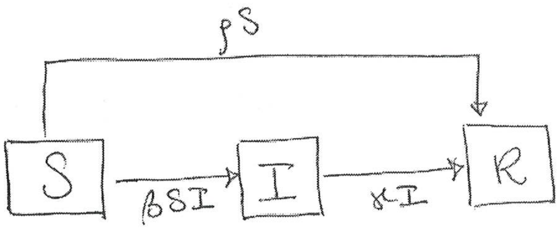
$\boxed{3}$ (c) How long after release does the mass pass through the equilibrium position? Include appropriate units.

$$y(t^*) = 0 \Leftrightarrow 5t^* + \frac{3\pi}{4} = n\pi \Leftrightarrow t^* = \left(n\pi - \frac{3\pi}{4}\right)\frac{1}{5}$$

minimum n for $t^* > 0$ is $n = 1$ so

$$t^* = \frac{\pi}{4}\frac{1}{5} = \frac{\pi}{20} \simeq 0.16 \text{ seconds}$$

5. Consider a disease for which immunity does not wane and for which there is a vaccine. We can draw the compartmental diagram for this disease as



Note that the total population N is constant, so $S + I + R = N$, and R includes people who are removed either by getting vaccinated, recovering from the disease, or dying from the disease. The parameter ρ is the vaccination rate.

$\boxed{3}$ (a) Write down the system of ODEs for this model.

$$\begin{cases} \dfrac{dS}{dt} = -\beta S I - \rho S \\ \dfrac{dI}{dt} = \beta S I - \gamma I \\ \dfrac{dR}{dt} = \gamma I + \rho S \end{cases}$$

$\boxed{7}$ (b) If we let $u = S/N$, $v = I/N$, $w = R/N$, and $\tau = \gamma\, t$, then we obtain the following two ODEs for u and v:

$$\frac{du}{d\tau} = \frac{-\beta N}{\gamma} uv - \frac{\rho}{\gamma} u, \quad \frac{dv}{d\tau} = \frac{\beta N}{\gamma} uv - v.$$

$= f(u,v)$ $\qquad$ $= g(u,v)$

Compute the Jacobian at the disease-free equilibrium (1,0). Under what conditions will an epidemic occur?

$$J = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}\Bigg|_{(1,0)} = \begin{bmatrix} -\frac{\beta N}{\gamma} v - \frac{\rho}{\gamma} & -\frac{\beta N}{\gamma} u \\ \frac{\beta N}{\gamma} v & \frac{\beta N}{\gamma} u - 1 \end{bmatrix}\Bigg|_{(1,0)}$$

$$= \begin{bmatrix} -\frac{\rho}{\gamma} & -\frac{\beta N}{\gamma} \\ 0 & \frac{\beta N}{\gamma} - 1 \end{bmatrix}$$

$\therefore \lambda_1 = -\frac{\rho}{\gamma} < 0$

$\lambda_2 = \frac{\beta N}{\gamma} - 1 > 0$ if $\frac{\beta N}{\gamma} > 1$

So an epidemic will occur if $\frac{\beta N}{\gamma} > 1$.

$\boxed{4}$ (c) Figure 1 shows (top plot) $I(t)$ for different values of ρ, and (bottom plot) peak and final size as functions of the vaccination rate ρ, for $R_0 = 3.0$ and $\gamma = 1$. Explain the results, i.e., explain why vaccination has the observed effect on the epidemic curve, peak size, and final size.

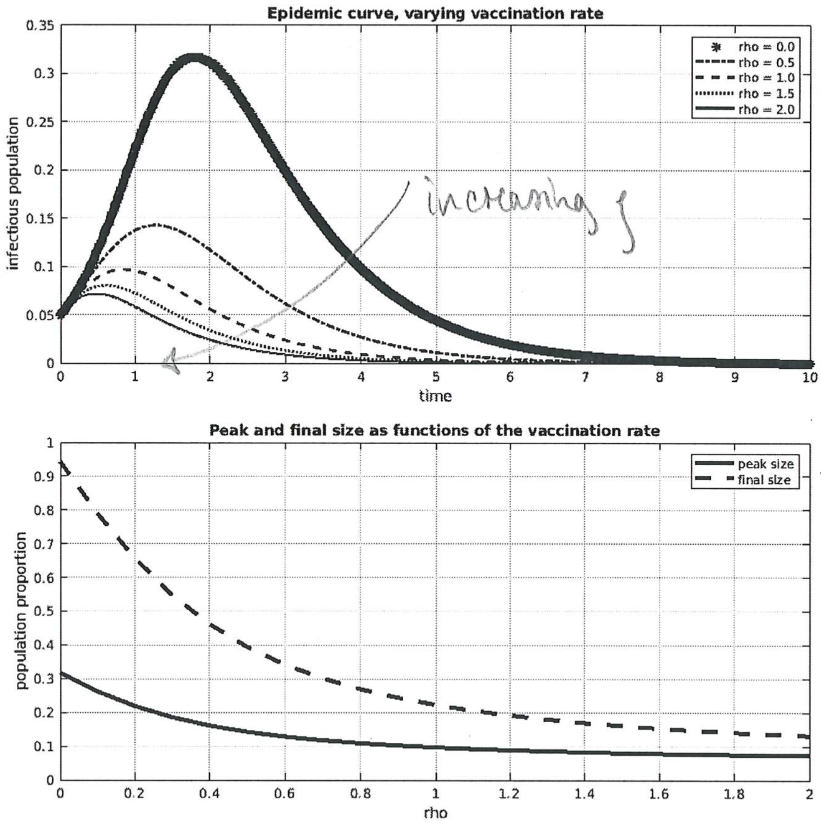


Figure 1: Output of the model.

From the top figure we see that as vaccination increases, the height <u>and</u> duration of the epidemic both decrease. This leads to the decrease in peak <u>and</u> final size seen in the bottom figure. These behaviours occur because vaccination removes individuals from the S class + puts them into the R class <u>bypassing</u> the I compartment, so there are fewer individuals to infect + fewer infectious individuals.

Question:	1	2	3	4	5	Total
Points:	4	4	10	16	14	48
Score:						