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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
 OF ARTS AND SCIENCES
 UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225

Date: April 17th, 2023 Time: 3:30pm Duration: 3 hours 30 minutes.

This exam has 10 questions for a total of 92 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	10	8	7	11	10	9	5	20	92
Score:											

1. Consider the first order autonomous ODE

$$\frac{dy}{dt} = y(y - a)(1 - y). \quad (1)$$

- 1 (a) Show that there are three equilibrium points.
- 3 (b) Sketch the phase line (including stability information) for the cases $0 < a < 1$ and $a > 1$. Explain how you decide the direction of the arrows.
- 2 (c) Suppose that (1) models a population of squeeets $S(t)$ that you are studying. Suppose also that currently $0 < a < 1$, but that climate change will increase a past 1 within the next half century. How do you therefore predict that climate change will affect the squeeet population?

- 3 2. (a) Given $y' = f(x, y)$, the value of y_{n+1} can be numerically predicted given a starting point (x_n, y_n) . Briefly explain how this is done for the Forward Euler (FE), Backward Euler (BE), and Heun methods by completing the sentences below.

• FE: y_{n+1} is found using the slope _____

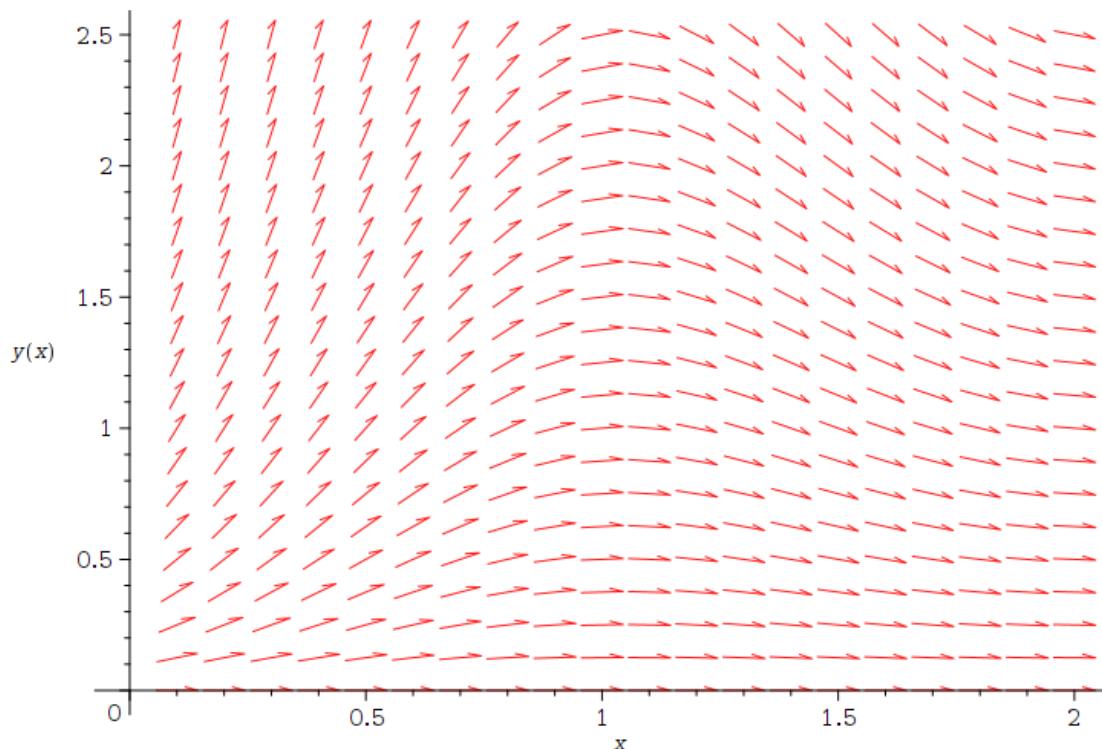
• BE: y_{n+1} is found using the slope _____

• Heun: y_{n+1} is found using the slope _____

- 3 (b) Consider the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} \sin(3x), \quad y(0.5) = 1. \quad (2)$$

Using $h = 0.5$ and the direction field below, carefully plot two steps of the solution to (2) using the Backward Euler, Forward Euler, and Heun methods. Make sure that it is clear which lines correspond to which method!



- 2 3. (a) Determine the most general function $M(x, y)$ for which the equation below is exact.

$$M(t, x)dt + (e^t - x)dx = 0$$

- 8 (b) Using the information above, solve the IVP below.

$$(xe^t + t)dt + (e^t - x)dx = 0, \quad x(0) = 1 \quad (3)$$

- 8 4. Suppose water containing salt at a concentration of 4 kg/L enters a tank of volume V that initially contains $x(0) = 0$ kg of salt. The rate of flow of brine into the tank is 5 L/min. The outflow rate is the same. If the mass of salt in the tank after 10 min is 156 kg, show that the volume of the tank is determined by the transcendental equation

$$e^{-50/V} = \frac{V - 39}{V}.$$

- 7 5. Consider the unforced mass-spring system whose motion is described by

$$x(t) = \frac{1}{4}e^{-2t}(\cos(3t) - \sin(3t)). \quad (4)$$

Suppose that $m = 1$. Determine the equation of motion (the ODE) for this mass-spring system, as well as the initial conditions at $t = 0$. All numerical values should be given as fractions, not decimals.

- 6. Consider the mass-spring system $y'' + by' + y = 0$.
- 3 (a) For what value of b is the system critically damped? Find the general solution in this case.
- 4 (b) Suppose now that the critically damped system is subject to forcing satisfying $f(t) = t^{-1}e^{-t}$. Show that the particular solution is given by $y_p(t) = t(\ln(t) - 1)e^{-t}$.
- 4 (c) With initial conditions $y(1) = 2e^{-1}$, $y'(1) = 5e^{-1}$, the system has one maximum at $t = t_m$. Show that t^* is given by the solution of $11 - 6t^* + (1 - t^*) \ln(t^*) = 0$.

7. Consider the mass-spring system given by

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = \cos(2t),$$

with fundamental solution set $\{e^{-t/2} \cos(\sqrt{3}/2), e^{-t/2} \sin(\sqrt{3}/2)\}$.

- 7 (a) Derive the steady-state behaviour using the Method of Undetermined Coefficients, and write the behaviour as a single phase-shifted sine.

- 3 (b) What is the gain factor for this system? Could it be increased? How? Be specific.

8. Consider an SIR-type disease where vaccination can provide permanent immunity. If a proportion p of the susceptible population is vaccinated, then the SIR model becomes

$$\frac{dS}{dt} = -\beta(1-p)SI \quad (5a)$$

$$\frac{dI}{dt} = \beta(1-p)SI - \gamma I \quad (5b)$$

$$\frac{dR}{dt} = \gamma I \quad (5c)$$

- 3 (a) Draw the compartmental diagram for this disease. What does the parameter β represent?
- 3 (b) What proportion of individuals need to be vaccinated to remove the threat of an epidemic? Assume that $R_0 = 5$ in the absence of vaccination (i.e., if $p = 0$).

- 3 (c) What assumption do we make about the population in using the compartmental modelling approach? If this assumption is not met, is the real p needed greater or smaller than the one you calculated? Why?

- 5 9. Prove property P2 (from the Brief Table of Laplace Transforms included with this test). Assume that $f(t)$ is continuous on $[0, \infty)$ and of exponential order.

10. Consider the initial value problem

$$y'' + 9y = g(t), \quad y(0) = y'(0) = 0, \quad (6)$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 2, \\ 3(t - 2), & 2 \leq t < 5, \\ 9, & t \geq 5. \end{cases} \quad (7)$$

- 1 (a) The function $g(t)$ is known as “ramp loading”. Sketch the function.
- 6 (b) Describe the qualitative nature of the solution that you expect from the IVP. Include a sketch. Explain your reasoning.

- 2 (c) Show that $g(t)$ can be expressed as

$$g(t) = 3(t - 2)H(t - 2) - 3(t - 5)H(t - 5). \quad (8)$$

- 8 (d) Solve the IVP (6) using the method of Laplace Transforms. Within your calculations, identify any properties you use by label (letter name (P#) or equation number - see the “Potentially Useful Information” provided with this test).

(Extra workspace for part d.)

- 3 (e) Write your solution $y(t)$ as piecewise continuous function (i.e., in the format used for (7)), and verify that it matches the behaviour you predicted in part (b).

Some Potentially Useful Information

BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$	e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}, \quad s > 0$	$\sin(bt)$	$\frac{b}{s^2 + b^2}, \quad s > 0$
$\cos(bt)$	$\frac{s}{s^2 + b^2}, \quad s > 0$	$e^{at}t^n, \quad n = 1, 2, \dots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$	$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
$\delta(t-a)$	$e^{-as}, \quad s > 0$	$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > a$

BRIEF TABLE OF PROPERTIES OF THE LAPLACE TRANSFORM

- P1 $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
 P2 $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
 P3 $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
 P4 $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$
 P5 $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n} (\mathcal{L}\{f\}(s))$

THEOREM: TRANSLATION IN t

Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > a \geq 0$. If a is a positive constant, then

$$\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s), \quad (9)$$

and, conversely, an inverse Laplace transform of $e^{-as}F(s)$ is given by

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)H(t-a), \quad (10)$$

where $H(t)$ is the Heaviside function. A corollary of equations (9) and (10) is the pair of equations

$$\mathcal{L}\{g(t)H(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s) \quad (11a)$$

$$g(t)H(t-a) = \mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{g(t+a)\}(s)\} \quad (11b)$$