UBC ID #: ______________ NAME (print): ______________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

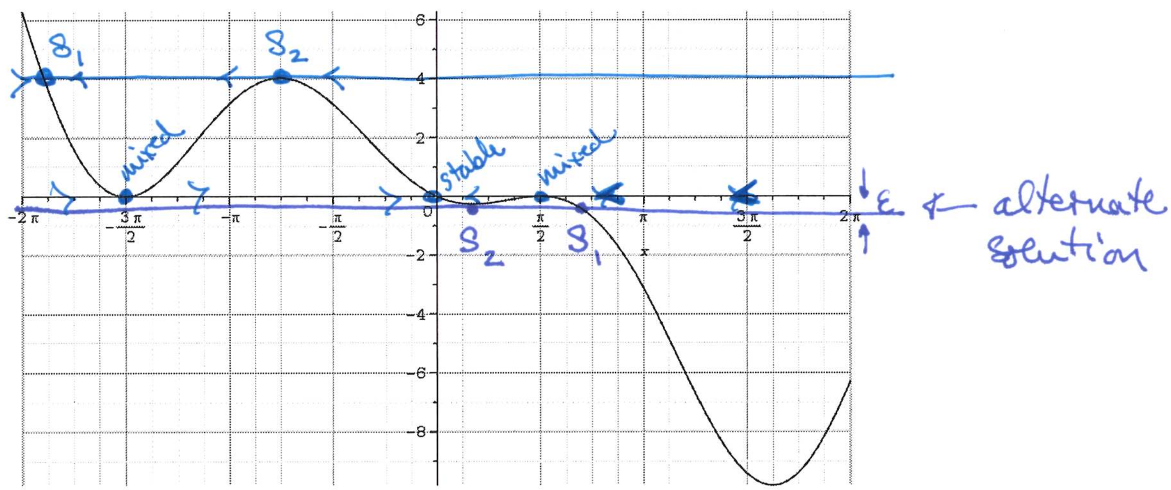
Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 6th, 2023 Time: 4:00pm Duration: 35 minutes.
This exam has 6 questions for a total of 24 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

1. The figure below is a plot of $f(x)$. Assume that outside the interval shown, the function never again crosses the horizontal axis. More specifically, the function is continuously increasing for $x < -3\pi/2$, peaks shortly to the right of $x = 2\pi$ and is continuously decreasing thereafter (for $x > 2\pi$).



$\boxed{3}$ (a) Use the horizontal axis (i.e., the line $f(x) = 0$) as your phase axis, and sketch the phase line for the ODE $x' = f(x)$. State the nature of the equilibria.

(b) Now imagine shifting the function $f(x)$ up or down by an arbitrary amount a.

$\boxed{2}$ i. What is the smallest shift size a at which the phase line has exactly two steady states? Specify if the shift is up or down. shift down by $a = 4$ (or up by ε)

$\boxed{2}$ ii. What are the two steady states and what is their stability? *Hint: You might find it useful to draw a horizontal line through the plot above, in the appropriate place, and indicate the steady states on that new line.*

The two new steady states are S_1 (stable) and S_2 (mixed)

$\boxed{4}$ 2. Solve the ODE

$$\frac{dy}{dx} + xy^2 = 0$$

Make sure you give *all* of the solutions!

$$\frac{dy}{dx} = -xy^2 \iff \boxed{y = 0} \text{ or } \frac{dy}{y^2} = -x\,dx \iff (1)$$

$$(1) \iff -\frac{1}{y} = -\frac{x^2}{2} + C$$

$$\iff +\frac{1}{y} = +\frac{x^2 + \tilde{C}}{2}$$

$$\iff \boxed{y = \frac{2}{x^2 + \tilde{C}}}$$

$\boxed{5}$ 3. Solve the ODE

$$x\frac{dy}{dx} + 3(y + x^2) = 1$$

$$\frac{x}{x}\frac{dy}{dx} + \frac{3y}{x} = \frac{1 - 3x^2}{x} \quad \text{linear in } y$$

$$\mu(x) = e^{\int 3/x\,dx} = e^{3\ln|x|} = e^{\ln|x^3|} = |x^3|$$

Choose $\mu(x) = x^3$. Then the ODE becomes:

$$\int \frac{d}{dx}\left[x^3 y\right] dx = \int \left(\frac{1}{x} - 3x\right) x^3\,dx \iff (11)$$

$$(11) \iff x^3 y = \int (x^2 - 3x^4)\,dx = \frac{x^3}{3} - \frac{3}{5}x^5 + C$$

$$\iff \boxed{y = \frac{1}{3} - \frac{3}{5}x^2 + \frac{C}{x^3}}$$

$\boxed{2}$ 4. Find the most general function $R(p,q)$ so that the equation below is exact.

$$R(p,q)dq + (q\cos(p) + e^q)dp = 0$$

$$\frac{\partial R}{\partial p} = \frac{\partial}{\partial q}\left(q\cos(p) + e^q\right) \iff$$

$$\iff \frac{\partial R}{\partial p} = \cos(p) + e^q \Rightarrow \boxed{R(p,q) = \sin(p) + pe^q + h(q)}$$

Where $h(q)$ is an arbitrary function of q.

$\boxed{2}$ 5. Set up the partial fraction decomposition (i.e. just set up the fractions - do not solve for the coefficients!) of

$$\frac{1}{1-x^4} = \frac{1}{(1-x)(1+x)(1+x^2)}. = F$$

$$F = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx + D}{1+x^2}$$

Where $A, B, C,$ & D are unknown constants.

6. Numerical solution of the ODE for $r(t)$ (not shown), using some unknown method, yields the results shown below.

stepsize	function value	difference
$h = 0.1$	$r(2) = 2.28835$	
$h = 0.05$	$r(2) = 2.26262$	-0.02573
$h = 0.025$	$r(2) = 2.24945$	-0.01317
$h = 0.0125$	$r(2) = 2.24279$	-0.00666 ← here the difference is less than 0.01
$h = 0.00625$	$r(2) = 2.23943$	-0.00336
$h = 0.003125$	$r(2) = 2.23775$	-0.00168
$h = 0.0015625$	$r(2) = 2.23691$	

$\boxed{2}$ (a) Why does the value of $r(2)$ keep changing?

As the stepsize is decreased, the method does a better job of sampling + tracking the direction field, so the predicted value of $r(2)$ keeps changing.

$\boxed{2}$ (b) Based on the information given, determine the value of $r(2)$ within two decimal places (± 0.01). Fill in the table as you do this, and explain how you arrived at your answer.

We see that as h decreases by half, the difference between successive approximations decreases, suggesting that the method is approaching the correct solution.

At $h = 0.0125$, the difference in approximations is less than the target, 0.01.

Taking two more halvings of h, we arrive at

$r(2) \simeq 2.23775 \pm 0.01$, or $r(2) = 2.24 \pm 0.01$

Question:	1	2	3	4	5	6	Total
Points:	7	4	5	2	2	4	24
Score:							