UBC ID #: ____________ NAME (print): ____________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

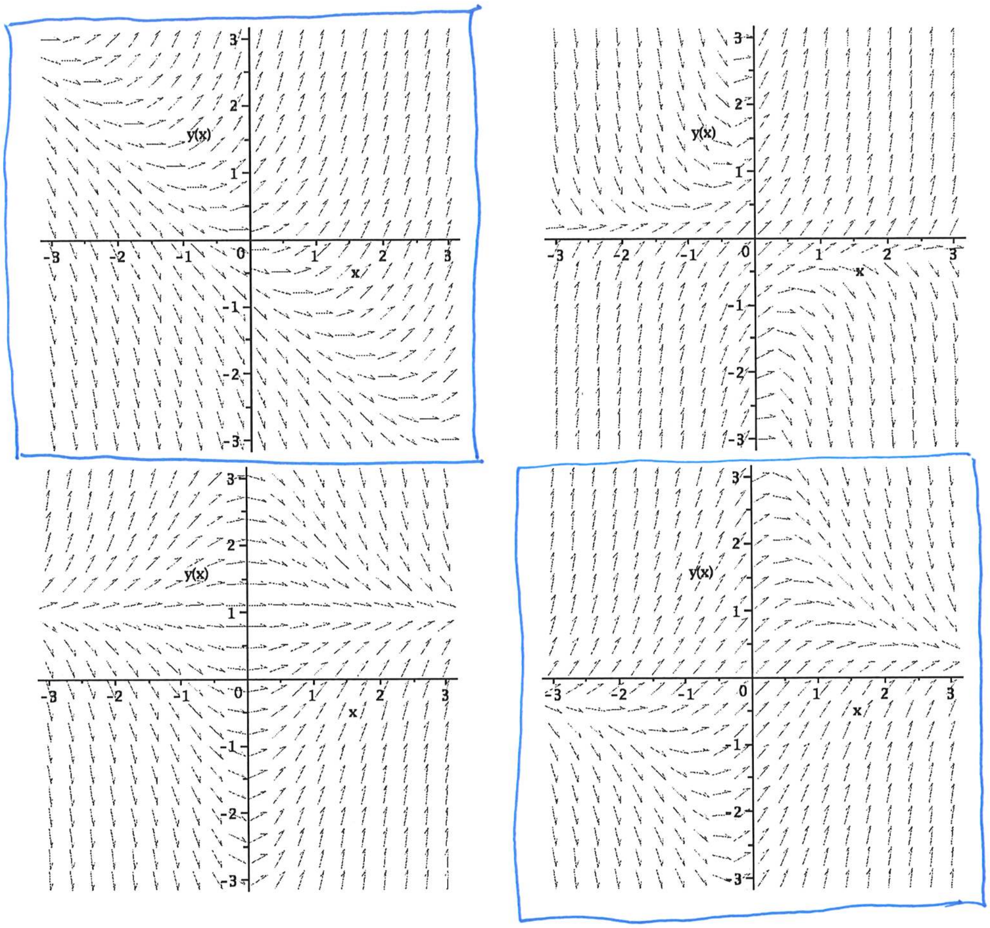
Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 3rd, 2023 Time: 4:00pm Duration: 35 minutes.
This exam has 5 questions for a total of 27 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

direction field for (b)

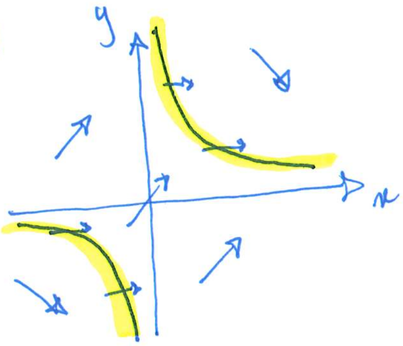


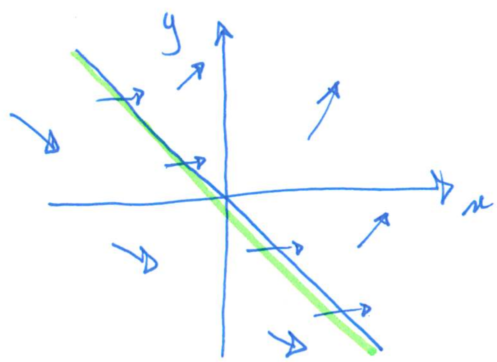
direction field for (a)

4 1. For each of the equations below, find and sketch the zero isocline (i.e. $dy/dx = 0$). Use that information to roughly fill in the direction field arrows in the rest of the (x, y) plane. Then match each of your direction field sketches with the correct direction field above.

$$a)\quad \frac{dy}{dx} = 1 - xy \qquad b)\quad \frac{dy}{dx} = x + y$$

a)





$\frac{dy}{dx} = 0 \Leftrightarrow 1 - xy = 0$

$\Leftrightarrow \boxed{y = \frac{1}{x}}$

$\frac{dy}{dx} = 0 \Leftrightarrow x + y = 0 \Leftrightarrow y = -x$

$\boxed{6}$ 2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1,$$

and determine the interval in which the solution exists.

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \iff \int 2(y-1)\,dy = \int (3x^2+4x+2)\,dx$$

$$\iff y^2 - 2y = x^3 + 2x^2 + 2x + C$$

Apply the IC:

$$y(0) = -1 \iff (-1)^2 - 2(-1) = 0^3 + 2(0)^2 + 2(0) + C$$

$$\iff C = 3$$

The solution is thus

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

The solution exists on $y \in (-\infty, 1)$ or $(1, \infty)$. ~~[illegible]~~ We need to convert this into an interval on x. ~~[illegible]~~

If $y = 1$ then we have

$$1^2 - 2(1) = x^3 + 2x^2 + 2x + 3 \iff x^3 + 2x^2 + 2x + 3 = 1 \iff x^3 + 2x^2 + 2x + 4 = 0$$

We see that $x = -2$ is a root. To find the other roots we divide:

$$\begin{array}{r|l} & x^2 + 0 + 2 \\ \hline x+2 & x^3 + 2x^2 + 2x + 4 \\ & x^3 + 2x^2 \\ \hline & 0 + 2x + 4 \\ & 2x + 4 \\ \hline & 0 \end{array}$$

So $x^3 + 2x^2 + 2x + 4 = (x+2)(x^2+2)$

The only real root is at $x = -2$, so the solution exists on $x \in (-2, \infty)$.

5 3. Show that the ODE below is not exact, then find an integrating factor of the form $x^m y^n$.

$$(3y^2 - 6xy)dx + (3xy - 4x^2)dy = 0$$

Let $\begin{cases} M(x,y) = 3y^2 - 6xy \\ N(x,y) = 3xy - 4x^2 \end{cases}$ then $\left.\begin{aligned} \frac{\partial M}{\partial y} &= 6y - 6x \\ \frac{\partial N}{\partial x} &= 3y - 8x \end{aligned}\right\} \neq$ So not exact.

Let $\mu(x,y) = x^m y^n$. Then

$$\frac{\partial}{\partial y}\left(x^m y^n (3y^2 - 6xy)\right) = \frac{\partial}{\partial x}\left(x^m y^n (3xy - 4x^2)\right) \Leftrightarrow (1)$$

$$(1) \Leftrightarrow \frac{\partial}{\partial y}\left(3x^m y^{2+n} - 6x^{m+1} y^{n+1}\right) = \frac{\partial}{\partial x}\left(3x^{m+1} y^{n+1} - 4x^{m+2} y^n\right)$$

$$\Leftrightarrow 3(2+n)x^m y^{n+1} - 6(n+1)x^{m+1} y^n = 3(m+1)x^m y^{n+1} - 4(m+2)x^{m+1} y^n$$

So $\begin{cases} 3(2+n) = 3(m+1) \\ -6(n+1) = -4(m+2) \end{cases} \Leftrightarrow \begin{cases} 2+n = m+1 \\ 3(n+1) = 2(m+2) \end{cases} \Leftrightarrow \begin{cases} m-n = 1 \\ 2m - 3n = -1 \end{cases} \Leftrightarrow \begin{cases} n = 3 \\ m = 4 \end{cases}$

$\therefore \mu(x,y) = x^4 y^3$

4 4. Write the Taylor Series expansion of the function $f(x) = \cos(2x)$ about the point $x = 0$ and up to order $\mathcal{O}(x^6)$. Simplify your fractions as much as possible (no decimals!!!).

$f(x) = \cos(2x)$ $f'(x) = -2\sin(2x)$ $f''(x) = -4\cos(2x)$

$f'''(x) = 8\sin(2x)$ $f^{iv}(x) = 16\cos(2x)$ $f^{v}(x) = -32\sin(2x)$

$$\therefore f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + f^{v}(0)\frac{x^5}{5!} + \mathcal{O}(x^6)$$

$$= 1 + 0x - 4\frac{x^2}{2} + 0 + 16\frac{x^4}{4!} + 0x^5 + \mathcal{O}(x^6)$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 + \mathcal{O}(x^6)$$

□ 5. Consider the initial value problem

$$\frac{dy}{dt} = \frac{-t}{y}, \qquad y(0) = 3. \tag{1}$$

The exact solution is $y(t) = \sqrt{9 - t^2}$.

[2] (a) Write the Forward Euler equation for (1).

$$y_{n+1} = y_n + h\left(\frac{-t_n}{y_n}\right) = y_n - \frac{h t_n}{y_n}$$

[4] (b) Write the Heun method equation for (1).

$$z_n = y_n - \frac{h t_n}{y_n}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(-\frac{t_n}{y_n} + \frac{-t_{n+1}}{y_n - \frac{h t_n}{y_n}}\right) = y_n - \frac{h}{2}\left(\frac{t_n}{y_n} + \frac{t_{n+1}}{y_n - \frac{h t_n}{y_n}}\right)$$

[2] (c) When Maple is asked to plot the solution to (1), it generates the following warning: *"Warning, plot may be incomplete, the following errors(s) were issued: cannot evaluate the solution further right of 3.0000002, probably a singularity."* Explain.

$\frac{dy}{dt}$ is undefined when $y = 0$. From the exact solution, we see that $y = 0$ corresponds to $t = 3$. So the numerical method runs into an infinite slope and stops working.