UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Mar 20th, 2023 Time: 4:00pm Duration: 35 minutes.
This exam has 5 questions for a total of 24 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

$\boxed{4}$ 1. Find the general solution of the differential equation $4y'' - 4y' + y = 0$.

$4r^2 - 4r + 1 = 0 \iff r = \frac{2 \pm \sqrt{4-4}}{4} = \frac{1}{2}$

$\therefore y(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$

$\boxed{3}$ 2. Consider a mass-spring system with $m = 10$, $b = 0.1$ and $k = 10$. At what forcing frequency γ will the response of the system be largest?

$M(\gamma)$ is largest when $\gamma = \gamma_r$

$\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{10}{10} - \frac{(0.1)^2}{2(10)^2}} = \sqrt{1 - \frac{0.01}{200}}$

$= \frac{1}{10}\sqrt{\frac{199.99}{2}}$

$\boxed{6}$ 3. Use variation of parameters to express the particular solution to the ODE below using integrals (do not attempt to solve these!). Then write the general solution. The fundamental solution set is $\{e^t, e^{-t}\}$.

$$y'' - y = \frac{1}{t}$$

Try $y_p(t) = v_1(t)e^t + v_2(t)e^{-t}$. Then

$\begin{cases} e^t v_1' + e^{-t} v_2' = 0 \\ e^t v_1' - e^{-t} v_2' = \frac{1}{t} \end{cases} \iff \begin{cases} 2e^t v_1' = \frac{1}{t} \\ 2e^{-t} v_2' = -\frac{1}{t} \end{cases} \iff \begin{cases} v_1' = \frac{e^{-t}}{2t} \\ v_2' = -\frac{e^t}{2t} \end{cases}$

$\therefore v_1 = \int \frac{e^{-t}}{2t}\,dt, \quad v_2 = -\int \frac{e^t}{2t}\,dt$

and so

$y(t) = \left[\int \frac{e^{-t}}{2t}\,dt\right] e^t - \left[\int \frac{e^t}{2t}\,dt\right] e^{-t} + c_1 e^t + c_2 e^{-t}$

4. A 1/4 kg mass is attached to a spring with stiffness 1 N/m, and damping constant 1/4 Ns/m. The mass is moved 1 m to the left of equilibrium and released.

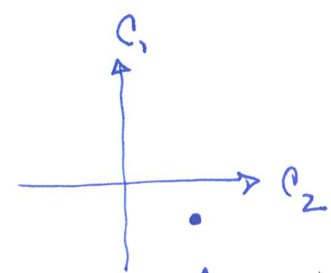
$\boxed{2}$ (a) Write down the differential equation and initial conditions that describe the motion of the mass.

$$\frac{1}{4}y'' + \frac{1}{4}y' + 1y = 0 \;;\quad y(0) = -1,\ y'(0) = 0$$

$\boxed{4}$ (b) Write the equation of motion (given below) as a single phase-shifted sine.

$$y(t) = -\,e^{-t/2}\left[\cos\left(\frac{\sqrt{15}}{2}t\right) - \frac{\sqrt{15}}{15}\sin\left(\frac{\sqrt{15}}{2}t\right)\right].$$

$$A = \sqrt{1^2 + \left(\frac{\sqrt{15}}{15}\right)^2} = \sqrt{\frac{15^2+15}{15^2}} = \sqrt{\frac{16(15)}{15^2}} = \frac{4\sqrt{15}}{15}$$



$$\tan\phi = \frac{-1}{\sqrt{15}/15} = \frac{-15}{\sqrt{15}} = -\sqrt{15}$$

so no need to add π

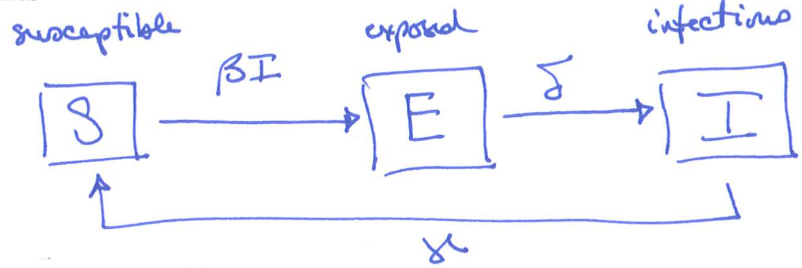
and $\phi = \arctan(-\sqrt{15})$

$$\therefore\ y(t) = e^{-t/2}\,\frac{4\sqrt{15}}{15}\sin\left(\frac{\sqrt{15}}{2}t + \arctan(-\sqrt{15})\right)$$

5. Consider the disease described by the ODEs below.

$$\frac{dS}{dt} = -\beta IS + \gamma I, \quad \frac{dE}{dt} = \beta IS - \delta E, \quad \frac{dI}{dt} = \delta E - \gamma I,$$

$\boxed{3}$ (a) Draw the compartmental diagram for the disease and state what the quantity $1/\delta$ represents



$\frac{1}{\delta}$ = mean time in E compartment

$\boxed{1}$ (b) Using $u = S/N$, $y = E/N$, $v = I/N$, and $\tau = \gamma t$ we arrive at the dimensionless system below. Check that $(1, 0, 0)$ is an equilibrium (DFE) for this system.

$$\frac{du}{d\tau} = -\frac{\beta N}{\gamma} uv + v, \quad \frac{dy}{d\tau} = \frac{\beta N}{\gamma} uv - \frac{\delta}{\gamma} y, \quad \frac{dv}{d\tau} = \frac{\delta}{\gamma} y - v,$$

@(1,0,0):

$\frac{du}{d\tau} = 0+0=0$, $\frac{dy}{d\tau} = 0-0=0$, $\frac{dv}{d\tau} = 0-0=0$, so yes, the point (1,0,0) is a steadystate (equilibrium).

$\boxed{2}$ (c) Using $u + y + v = 1$, $a = \delta/\gamma$, and $R = \beta N/\gamma$, we can reduce the dimensionless system to the 2nd order system below. Write the Jacobian for this system, and then plug in the DFE.

$$\frac{du}{d\tau} = -Ruv + v, \quad \frac{dv}{d\tau} = a(1 - u + v) - v.$$

$= f(u,v)$ $\qquad = g(u,v)$

$$J = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} = \begin{bmatrix} -Rv & -Ru+1 \\ -a & a-1 \end{bmatrix}$$

$$\text{So } J_{DFE} = \begin{bmatrix} 0 & 1-R \\ -a & a-1 \end{bmatrix}$$

$\boxed{2}$ (d) The eigenvalues of the Jacobian at the DFE are given below. Assume that $0 < a < 1$, and explain why $R_0 = R$ in this case.

$$\lambda_{1,2} = \frac{1}{2}\left[(a-1) \pm \sqrt{(a-1)^2 - 4(1-R)a}\right].$$

If $0<a<1$ then $a-1<0$. So λ_2 always has negative real part. In order for λ_1 to have positive real part, we require

$4(1-R)a < 0 \Leftrightarrow 1-R<0 \Leftrightarrow R>1$

The threshold value for an epidemic is the value at which λ_1 becomes positive, so we have

$R_0 = R$

Question:	1	2	3	4	5	Total
Points:	4	3	6	6	8	27
Score:						