UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 12th, 2024 Time: 8:00am Duration: 35 minutes.
This exam has 5 questions for a total of 32 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

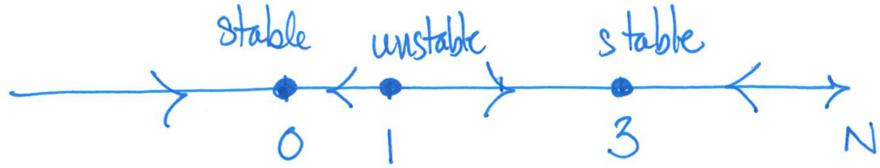
1. The growth of a certain population $N(t)$ is described by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{3}\right)(N-1), \tag{1}$$

where $r > 0$ and $t \geq 0$.

$\boxed{5}$ (a) Sketch the phase line. Label the steady states with their stability.

Roots at $N=0$, $N=3$, $N=1$



$\boxed{2}$ (b) Suppose $N(0) > 0$. What is $\lim_{t\to\infty} N(t)$?

If $N(0) < 1$, then $\lim_{t\to\infty} N(t) = 0$. If $N(0) = 1$, then $\lim_{t\to\infty} N(t) = 1$.
If $N(0) > 1$, then $\lim_{t\to\infty} N(t) = 3$.

$\boxed{6}$ 2. Solve the IVP

$$y'' + 2y' + y = 0, \qquad y(0) = 0, \quad y'(0) = 3.$$

characteristic eqn: $r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0 \iff r = -1$

$\therefore\ y(t) = c_1 e^{-t} + c_2 t e^{-t}$

apply the ICs:

$y(0) = 0 \iff c_1 = 0$

$y'(0) = 3 \iff c_2 e^{-t}\Big|_{t=0} - c_2 t e^{-t}\Big|_{t=0} = 3 \iff c_2 = 3$

$\therefore\ \boxed{y(t) = 3te^{-t}}$

5 3. Consider the IVP

$$t\frac{dy}{dt} - y = t^2\sin(t), \qquad y(\pi) = 0, \quad t \geq 0.$$

The ODE is linear in y. Use this information to solve the IVP.

If $t = 0$ then we have $y = 0$.

If $t > 0$ then we have, in standard form

$$\frac{dy}{dt} - \frac{1}{t}y = t\sin(t)$$

We seek an integrating factor

$$\mu(t) = e^{-\int \frac{1}{t}dt} = e^{-\ln(t) + \cancel{c}^{0}} = e^{\ln\left(\frac{1}{t}\right)} = \frac{1}{t}$$

$\therefore$ the ODE becomes

$$\frac{1}{t}\cdot\frac{dy}{dt} - \frac{1}{t^2}y = \sin(t) \Leftrightarrow \frac{d}{dt}\left(\frac{1}{t}y\right) = \sin(t) \quad \int$$

$$\int \Leftrightarrow \frac{1}{t}y = -\cos(t) + C \Leftrightarrow y(t) = -t\cos(t) + Ct$$

Now apply the IC:

$$y(\pi) = 0 \Leftrightarrow 0 = -\pi\cos(\pi) + C\pi \Leftrightarrow 0 = \pi + C\pi$$

$$\Leftrightarrow C = -1$$

$$\therefore \boxed{y(t) = -t(\cos(t) + 1)}$$

6 4. Verify that the ODE below is exact, and solve it.

$$\underbrace{(e^{x+y} + 2x)}_{M(x,y)}dx + \underbrace{(e^{x+y} - 2y)}_{N(x,y)}dy = 0$$

Test:

$\frac{\partial M}{\partial y} = e^{x+y}$ and $\frac{\partial N}{\partial x} = e^{x+y}$ So the ODE is exact.

Integrating with respect to x we obtain

$$F(x,y) = \int (e^{x+y} + 2x)\,dx = e^{x+y} + x^2 + h(y)$$

Differentiating with respect to y we obtain

$$\frac{\partial F}{\partial y} = N(x,y) \iff e^{x+y} + \frac{dh}{dy} = e^{x+y} - 2y \iff \frac{dh}{dy} = -2y$$

$$\iff h(y) = -y^2 + C$$

$$\therefore F(x,y) = e^{x+y} + x^2 - y^2 + C$$

The solutions are the level curves of F:

$$\boxed{e^{x+y} + x^2 - y^2 = K}$$

5. Consider the ODE:

$$\frac{dx}{dt} = \frac{1}{x}. \tag{2}$$

$\boxed{2}$ (a) Write down the formula for the Forward Euler approximation with stepsize h.

$$x_{n+1} = x_n + h\frac{1}{x_n} \quad \cdots\cdots (1)$$

$\boxed{3}$ (b) Write down the formula for the Heun approximation with stepsize h.

$$z_n = x_n + h\frac{1}{x_n}$$

$$x_{n+1} = x_n + \frac{h}{2}\left(\frac{1}{x_n} + \frac{1}{z_n}\right)$$

$\boxed{3}$ (c) The Taylor Series expansion of $x(t_n + h)$ around $x(t_n) = x_n$ is

$$x(t_n + h) = x_n + h\left.\frac{dx}{dt}\right|_{x_n} + \frac{h^2}{2}\left.\frac{d^2x}{dt}\right|_{x_n} + O(h^3).$$

Rewrite the derivatives (dx/dt and d^2x/dt^2) in the expansion in terms of $f(x) = 1/x$, and then compare the expansion with the **Forward Euler** formula you gave in part (a). How are they similar? What is the meaning of the difference?

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\text{So } x(t_n+h) = x_n + h\frac{1}{x_n} - \frac{h^2}{2}\frac{1}{x_n^2} + O(h^3) \quad \cdots (2)$$

Similarities btw (1) + (2):

The two are the same until the $O(h^2)$ term, so we conclude that the FE method is $O(h)$ (i.e., it is a first-order method).