UBC ID #: ______________ NAME (print): ______________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Feb 9th, 2024 Time: 8:00am Duration: 35 minutes.
This exam has 6 questions for a total of 27 points.

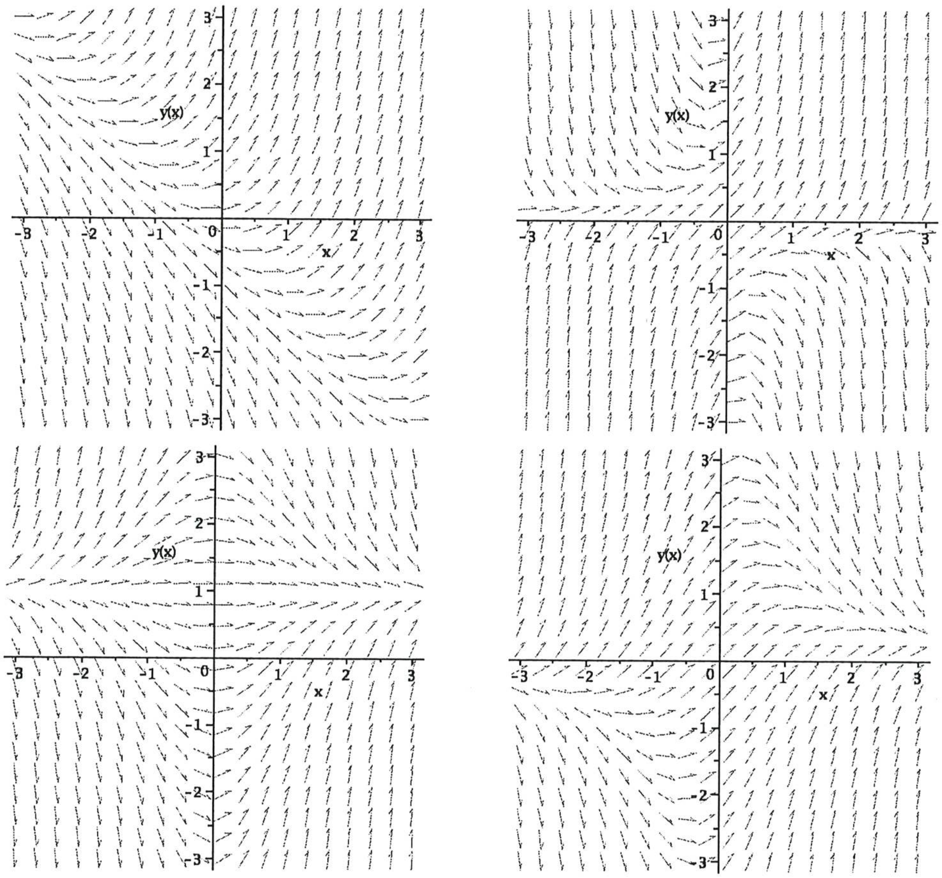
SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Question:	1	2	3	4	5	6	Total
Points:	4	6	5	4	4	4	27
Score:							

$\frac{dy}{dx} = x+y$



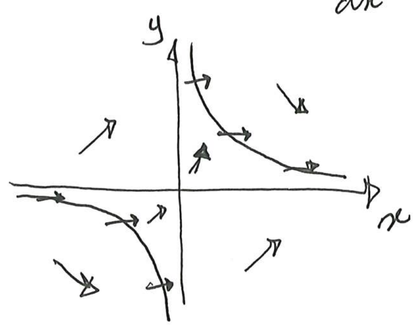
$\frac{dy}{dx} = 1-xy$

$\boxed{4}$ 1. For each of the equations below, find and sketch the zero isocline (i.e. $dy/dx = 0$). Use that information to roughly fill in the direction field arrows in the rest of the (x, y) plane. Then match each of your direction field sketches with the correct direction field above.

a) $\frac{dy}{dx} = 1 - xy$ b) $\frac{dy}{dx} = x + y$

$\frac{dy}{dx} = 0 \iff 1 - xy = 0 \iff xy = 1$

So $y = \frac{1}{x}$, $x \neq 0$ is the zero isocline. Also, $\frac{dy}{dx} > 0$ if xy is small.

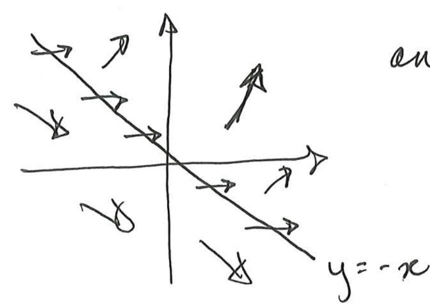


This looks like the direction field on the bottom right.

$\frac{dy}{dx} = 0 \iff x + y = 0 \iff y = -x$

So $y = -x$ is the zero isocline.

Also, $\frac{dy}{dx} > 0$ if x and y are positive.



This looks like the direction field on the top left.

$\boxed{6}$ 2. Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1,$$

and determine the interval in which the solution exists.

The ODE is separable. So we write

$$\int 2(y-1)\,dy = \int (3x^2+4x+2)\,dx \iff y^2 - 2y = x^3 + 2x^2 + 2x + C$$

Now apply the IC

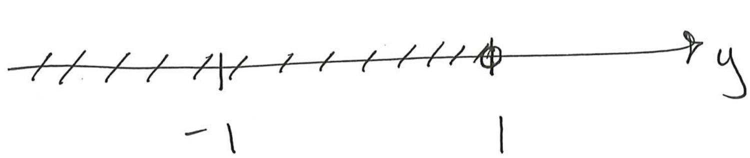
$$y(0) = -1 \iff (-1)^2 - 2(-1) = 0 + 0 + 0 + C$$

$$\iff 1 + 2 = C \iff C = 3$$

$\therefore$ the implicit solution is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

and the solution is valid on any interval containing the point $(0,-1) = (x,y)$ and for which $y - 1 \neq 0 \iff y \neq 1$



So the solution is valid on $-\infty < y < 1$

$\boxed{5}$ 3. Show that the ODE below is not exact, then find an integrating factor of the form $x^m y^n$.

$$\underbrace{(3y^2 - 6xy)}_{M(x,y)}dx + \underbrace{(3xy - 4x^2)}_{N(x,y)}dy = 0$$

$\dfrac{\partial M}{\partial y} = 6y - 6x \qquad \dfrac{\partial N}{\partial x} = 3y - 8x \qquad \because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the ODE is not exact.

We seek an integrating factor $\mu(x,y) = x^m y^n$. Multiplying the ODE by μ, we obtain

$$\underbrace{x^m y^n (3y^2 - 6xy)}_{\bar{M}}dx + \underbrace{x^m y^n (3xy - 4x^2)}_{\bar{N}}dy = 0$$

For exactness, we require $\dfrac{\partial \bar{M}}{\partial y} = \dfrac{\partial \bar{N}}{\partial x} \iff 1/$

$1/ \iff 3x^m(2+n)y^{n+1} - 6x^{m+1}(n+1)y^n + 3(m+1)x^m y^{n+1} - 4(2+m)x^{m+1}y^n = 0$

$$\iff \begin{cases} 3(2+n) = 3(m+1) \\ -6(n+1) = -4(2+m) \end{cases} \iff \begin{cases} m = n+1 \\ -4m = -6n+2 \end{cases} \iff \begin{cases} m = n+1 \\ -4(n+1) = -6n+2 \end{cases} \iff \begin{cases} m = n+1 \\ 2n = 6 \end{cases} \iff \begin{cases} m = 4 \\ n = 3 \end{cases}$$

$\therefore \mu(x,y) = x^4 y^3$

$\boxed{4}$ 4. Write the Taylor Series expansion of the function $f(x) = \cos(2x)$ about the point $x = 0$ and up to order $\mathcal{O}(x^6)$. Simplify your fractions as much as possible (no decimals!!!).

$f'(x) = -2\sin(2x) \quad f''(x) = -4\cos(2x) \quad f'''(x) = 8\sin(2x) \quad f^{IV}(x) = 16\cos(2x)$

$f^{V}(x) = -32\sin(2x)$, and $\cos(0) = 1$ and $\sin(0) = 0$

$$\therefore \cos(2x) = 1 - 4\frac{x^2}{2} + 16\frac{x^4}{4!} + \mathcal{O}(x^6)$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 + \mathcal{O}(x^6)$$

☐ 5. Consider the initial value problem

$$\frac{dy}{dt} = \frac{-t}{y}, \qquad y(0) = 3. \tag{1}$$

The exact solution is $y(t) = \sqrt{9 - t^2}$.

$\boxed{2}$ (a) Write the Forward Euler and Backward Euler equations for (1).

FE: $y_{n+1} = y_n + h\left(\frac{-t_n}{y_n}\right)$

BE: $y_{n+1} = y_n + h\left(-\frac{t_{n+1}}{y_{n+1}}\right)$

$\boxed{2}$ (b) When Maple is asked to plot the solution to (1), it generates the following warning: *"Warning, plot may be incomplete, the following errors(s) were issued: cannot evaluate the solution further right of 3.0000002, probably a singularity."* Explain.

$\because$ $y(0) > 0$ and $\frac{dy}{dt} < 0$ for $t > 0, y > 0$, we see that, at least initially, y will decrease. If it decreases to $y = 0$ however, $\frac{dy}{dt}$ becomes infinite, and the numerical solver is unable to handle this situation.

$\boxed{4}$ 6. Find the general solution of the differential equation $4y'' - 4y' + y = 0$.

The characteristic equation is

$4r^2 - 4r + 1 = 0 \iff (2r-1)^2 = 0 \iff r = \frac{1}{2}$

$\therefore y(x) = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$