UBC ID #: ____________ NAME (print): ____________________

Signature: Solutions

a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 225
Date: Mar 25th, 2024 Time: 8:00am Duration: 35 minutes.
This exam has 5 questions for a total of 35 points.

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. **Answers without accompanying work are worth zero.** Simplify all answers.
- The use of a calculator is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 35 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

$\boxed{2}$ 1. Consider the mass-spring system

$$my'' + by' + ky = 0. \tag{1}$$

Assume that all of the parameters are non-negative. Under what conditions is the solution of (1) considered to be underdamped?

char eqn: $mr^2 + br + k = 0$ $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

If the roots r are complex, i.e., $b^2 - 4mk < 0$, then the system (1) is underdamped.

$\boxed{3}$ 2. Consider the initial value problem

$$y'' + 0.1y' + 25y = 2\cos(\gamma t), \qquad y(0) = 1, \quad y'(0) = 0. \tag{2}$$

For which integer value of γ will the particular solution have the largest magnitude? What is this frequency called?

γ_r = resonance frequency

$$= \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} = \sqrt{\frac{25}{1} - \frac{(0.1)^2}{2 \cdot 1^2}}$$

$$= \sqrt{25 - \frac{1}{200}} = \sqrt{\frac{5000 - 1}{20}}$$

$$= \sqrt{\frac{4999}{20}}$$

3. Consider the equation

$$y'' + 25y = \cos(5t). \qquad (3)$$

$\boxed{5}$ (a) Use the method of undetermined coefficients to find a particular solution.

Hom Sol'n:

$r^2 + 25 = 0 \Leftrightarrow r = \pm 5i \quad \therefore y_h = c_1\cos(5t) + c_2\sin(5t)$

Part Sol'n:

try $y_p(t) = At\cos(5t) + Bt\sin(5t)$

$y_p'(t) = -5At\sin(5t) + 5Bt\cos(5t) + A\cos(5t) + B\sin(5t)$

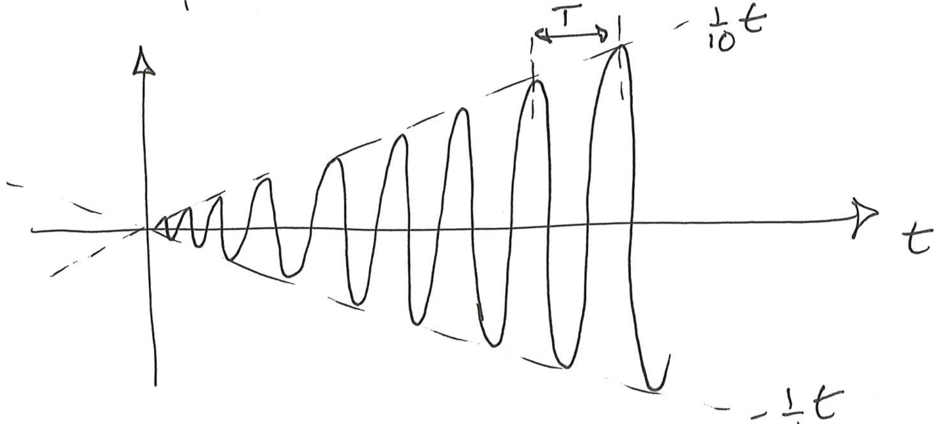
$y_p''(t) = -25At\cos(5t) - 25Bt\sin(5t) - 10A\sin(5t) + 10B\cos(5t)$

$\therefore y_p'' + 25y_p = \cos(5t) \Leftrightarrow -10A\sin(5t) + 10B\cos(5t) = \cos(5t)$

$\therefore \begin{cases} A = 0 \\ 10B = 1 \end{cases} \Leftrightarrow \begin{cases} A = 0 \\ B = \frac{1}{10} \end{cases}$

$\therefore y_p(t) = \frac{1}{10}t\sin(5t)$

$\boxed{3}$ (b) Sketch the particular solution & give the quasiperiod.



$T = \frac{2\pi}{5}$

$\boxed{8}$ 4. Use the method of variation of parameters to find a general solution to the differential equation

$$y'' + 2y' + y = e^{-t}. \tag{4}$$

Be sure to work from the system of two constraints on $v_1'(t)$ and $v_2'(t)$.

Hom Sol'n

$r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0 \iff r = -1$

$\therefore y_h(t) = c_1 e^{-t} + c_2 t e^{-t}$

Part sol'n

try $y_p(t) = v_1(t) e^{-t} + v_2(t) t e^{-t}$

then $\begin{cases} v_1' e^{-t} + v_2' t e^{-t} = 0 \\ -v_1' e^{-t} + v_2'(e^{-t} - t e^{-t}) = e^{-t} \end{cases} \iff /1/$

$/1/ \iff \begin{cases} v_1' + t v_2' = 0 \\ -v_1' + (1-t) v_2' = 1 \end{cases} \iff \begin{cases} v_2' = 1 \\ v_1' = -t v_2' \end{cases} \iff \begin{cases} v_2' = t \\ v_1' = -t \end{cases}$

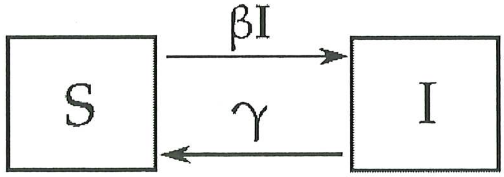
$\iff \begin{cases} v_2 = t \\ v_1 = -\frac{t^2}{2} \end{cases}$

$\therefore y_p(t) = -\frac{t^2}{2} e^{-t} + t^2 e^{-t} = \frac{t^2}{2} e^{-t}$

Gen'l Sol'n

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{t^2}{2} e^{-t}$

5. Consider the SIS disease diagram below. Assume also that $S + I = N$, a constant.



[2] (a) Write the ODEs for this model.

$$\begin{cases} \dfrac{dS}{dt} = -\beta I S + \gamma I \\ \dfrac{dI}{dt} = \beta I S - \gamma I \end{cases} \quad \dots \quad (1)$$

[5] (b) Non-dimensionalise the model (use $u = S/N$, $v = I/N$, $\tau = \gamma t$). Identify R_0.

Let $u = \dfrac{S}{N}$, $v = \dfrac{I}{N}$, $\tau = \gamma t$. Then (1) becomes

$$\frac{d(uN)}{d(\tau/\gamma)} = -\beta(vN)(uN) + \gamma(vN) \iff$$

$$\iff N\gamma \frac{du}{d\tau} = -\beta N^2 uv + N\gamma v$$

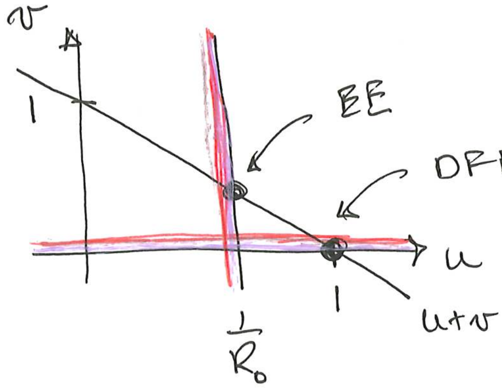
$$\iff \frac{du}{d\tau} = -\frac{\beta N}{\gamma} uv + v \qquad \text{where } R_0 = \frac{\beta N}{\gamma}$$

and $\dfrac{dv}{d\tau} = \dfrac{\beta N}{\gamma} uv - v$

(c) Write the population constraint in terms of u and v:

$$u + v = 1$$

9 (c) In the phase plane, sketch the nullclines and locate the steady states in the case $R_0 > 1$ (use the dimensionless equations). What is each steady state called? Give the coordinates of each one.



If $R_0 > 1$ then

$\frac{1}{R_0} < 1$.

Nullclines:

$$\begin{cases} \frac{du}{d\tau} = 0 \\ \frac{dv}{d\tau} = 0 \end{cases} \iff \begin{cases} (-R_0 u + 1) v = 0 \\ (R_0 u - 1) v = 0 \end{cases} \iff \begin{cases} u = \frac{1}{R_0} \text{ or } v = 0 \\ u = \frac{1}{R_0} \text{ or } v = 0 \end{cases}$$

So both derivatives are zero everywhere along both nullclines.

Two steady states:

Disease-free equilibrium (DFE), $(u,v) = (1,0)$

Endemic equilibrium (EE), $(u,v) = \left(\frac{1}{R_0}, 1 - \frac{1}{R_0}\right)$

Question:	1	2	3	4	5	Total
Points:	2	3	8	8	16	37
Score:						