A#2
Solutions

1. (G 10.2 #12)

$$\begin{cases} y'' + \lambda y = 0 & 0 < x < \pi/2 \\ y'(0) = 0,\ y'(\pi/2) = 0 & \text{(Neumann BCs)} \end{cases}$$

We first consider the ODE alone.

Case 1 : $\lambda = 0$

$y'' = 0 \iff y' = a \iff y = ax + b$

apply $\begin{cases} y'(0) = 0 \\ y'(\pi/2) = 0 \end{cases} \iff \begin{cases} a = 0 \\ a = 0 \end{cases}$

$\therefore y = b$ (a cst) is a solution and $\lambda = 0$ is an eigenvalue.

Case 2 : $\lambda > 0$ let $\lambda = \rho^2$

$y'' + \rho^2 y = 0$ has char. eqn. $r^2 + \rho^2 = 0 \iff r = \pm i\rho$

$\therefore y(x) = c_1 \cos(\rho x) + c_2 \sin(\rho x)$

apply $\begin{cases} y'(0) = 0 \\ y'(\pi/2) = 0 \end{cases} \iff \begin{cases} \cancel{c_1\rho \sin(\rho \cdot 0)}^{\,0} + c_2 \rho \cos(\rho \cdot 0) = 0 \\ c_1 \rho \sin(\rho \cdot \frac{\pi}{2}) + c_2 \rho \cos(\rho \cdot \frac{\pi}{2}) = 0 \end{cases}$

$\iff \begin{cases} c_2 = 0 \\ \rho = 2n \end{cases}$

$\therefore$ $\lambda_n = 4n^2$ is an eigenvalue for each n

$\cos(2nx)$ is ~~the~~ the corresponding eigenfunction,

$n = 1, 2, 3, \ldots$

Case 3: $\lambda < 0$ let $\lambda = -\rho^2$

$y'' - \rho^2 y = 0$ has char. eqn. $r^2 - \rho^2 = 0 \Rightarrow r = \pm\rho$

$\therefore y(x) = c_1 \cosh(\rho x) + c_2 \sinh(\rho x)$

apply $\begin{cases} y'(0) = 0 \\ y'\left(\frac{\pi}{2}\right) = 0 \end{cases} \Rightarrow \begin{cases} c_1 \rho \cancelto{0}{\sinh(\rho \cdot 0)} + c_2 \cosh(\rho \cdot 0) = 0 \\ c_1 \rho \sinh\left(\rho \cdot \frac{\pi}{2}\right) + c_2 \cosh\left(\rho \cdot \frac{\pi}{2}\right) = 0 \end{cases}$

$\Rightarrow \begin{cases} c_2 = 0 \\ c_1 = 0 \quad (\because \sinh\left(\frac{\rho\pi}{2}\right) \neq 0 \\ \qquad\qquad \text{for } \rho \neq 0\ ) \end{cases}$

$\therefore$ we only have trivial solutions in this case.

In summary, the eigenvalues & eigenfunctions of this BVP are

$\begin{cases} \lambda = 0, \quad y(x) = b \ (\text{a cst}) \\ \lambda = 4n^2, \quad y_n(x) = \cos(2nx) \quad , n = 1, 2, 3, \ldots \end{cases}$

2. ($\S$ 10.2 #14)

$$\begin{cases} y'' - 2y' + \lambda y = 0; & 0 < x < \pi \\ y(0) = 0, \ y(\pi) = 0 \end{cases}$$

The characteristic equation for the ODE is

$r^2 - 2r + \lambda = 0 \iff r = +1 \pm \sqrt{1-\lambda}$

Case 1: $\lambda = 1$

$r = 1$, double root, and the general solution is

$y(x) = c_1 e^x + c_2 x e^x$

apply $\begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 \pi e^{\pi} = 0 \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$

$\therefore$ we only have trivial solutions in this case

Case 2: $\lambda < 1$ let $1 - \lambda = +\rho^2$

$r = 1 \pm \sqrt{1-\lambda} = 1 \pm \rho$, two real, distinct, roots

$\therefore y(x) = \bar{c}_1 e^{(1+\rho)x} + \bar{c}_2 e^{(1-\rho)x}$

OR

$= e^x \left(c_1 \cosh(\rho x) + c_2 \sinh(\rho x) \right)$

apply $\begin{cases} y(0)=0 \\ y(\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 e^{\pi} \sinh(\rho\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1=0 \\ c_2=0 \text{ or } \\ \rho=0 \end{cases}$

If $\rho=0$, $\lambda=1$, which is not possible, so we have $c_2=0$ and only trivial solutions.

Case 3: $\lambda > 1$ let $1-\lambda = -\rho^2$

$r = 1 \pm \sqrt{1-\lambda} = 1 \pm i\rho$, two complex roots

$\therefore y(x) = e^{x}\left(c_1 \cos(\rho x) + c_2 \sin(\rho x)\right)$

apply $\begin{cases} y(0)=0 \\ y(\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 e^{\pi} \sin(\rho\pi)=0 \end{cases}$

$\Leftrightarrow \begin{cases} c_1=0 \\ c_2=0 \text{ or } \rho\pi = n\pi \end{cases}$

$\therefore$ we have the eigenvalues

$\lambda = 1+\rho^2 = 1+n^2$, $n=1,2,3,\dots$

and corresponding eigenfunctions

$y_n(x) = e^{x}\sin(nx)$

3. (§10.2 #18)

$$\begin{cases} \dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2}, & 0 < x \pi,\ t > 0 \quad \cdots \ (1) \\ u(0,t) = u(\pi,t) = 0, & t > 0 \quad \cdots \ (2) \\ u(x,0) = \sin(4x) + 3\sin(6x) - \sin(10x), & 0 < x < \pi \quad \cdots \ (3) \end{cases}$$

Step 1: Apply Sep'n of Vars

let $u(x,t) = F(x)G(t)$, then the PDE becomes

$$F(x)G'(t) = 3F''(x)G(t) \Leftrightarrow 1/$$

$$1/ \Leftrightarrow \frac{G'(t)}{3G(t)} = \frac{F''(x)}{F(x)} = -\lambda$$

Step 2: Form the ODE problems

applying separation of variables to (2) we obtain

$$\begin{cases} F(0)G(t) = 0 \\ F(\pi)G(t) = 0 \end{cases}$$

For nontrivial solutions we require $\begin{cases} F(0) = 0 \\ F(\pi) = 0 \end{cases}$

$\therefore$ the ODE problems are

(A) $F''(x) + \lambda F(x) = 0$ with $F(0) = F(\pi) = 0$

(B) $G'(t) + 3\lambda G(t) = 0$

Step 3

(A) is a BVP. We solve it by considering 3 cases. The char equ for (A) is

$$r^2 + \lambda = 0 \iff r = \pm\sqrt{-\lambda}$$

Case 1: $\lambda = 0$

then r is a double real root and the general solution is

$$F(x) = ax + b$$

apply $\begin{cases} F(0) = 0 \\ F(\pi) = 0 \end{cases} \iff \begin{cases} b = 0 \\ a \cdot \pi = 0 \end{cases} \iff \begin{cases} b = 0 \\ a = 0 \end{cases}$

and we only have trivial solutions in this case.

Case 2: $\lambda < 0$ let $\lambda = -\rho^2$

then $r = \pm \rho$, two real roots, and the general solution is

$$F(x) = \bar{c}_1 e^{\rho x} + c_2 e^{-\rho x}$$

or

$$= c_1 \cosh(\rho x) + c_2 \sinh(\rho x)$$

apply $\begin{cases} F(0)=0 \\ F(\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1=0 \\ c_2 \sinh(\rho\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1=0 \\ c_2=0 \end{cases}$

and we only have trivial solutions in this case.

<u>Case 3: $\lambda > 0$</u> let $\lambda = \rho^2$

then $r = \pm i\rho$, complex roots (actually, pure imaginary) and the general solution is

$$F(x) = c_1 \cos(\rho x) + c_2 \sin(\rho x)$$

apply $\begin{cases} F(0)=0 \\ F(\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1=0 \\ c_2 \sin(\rho\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1=0 \\ c_2=0 \text{ or} \\ \rho=n \end{cases}$

For nontrivial solutions, we have eigenvalues $\lambda_n = \rho^2 = n^2$ and eigenfunctions $F_n(x) = \sin(nx)$.

(B) The corresponding solution to problem (B) is

$$G_n'(t) - 3n^2 G_n(t) = 0 \Leftrightarrow G_n(t) = a_n e^{-3n^2 t}$$

Step 4

The general solution is thus

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x)\, G_n(t)$$

$$= \sum_{n=1}^{\infty} a_n e^{-3n^2 t} \sin(nx)$$

Step 5

Applying the IC

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = \sin(4x) + 3\sin(6x) - \sin(10x)$$

we obtain ($\because$ $\sin(nx)$ & $\sin(mx)$ are linearly independent functions $\forall\, n \neq m$)

$$\begin{cases} a_4 = 1, \quad a_6 = 3, \quad a_{10} = -1 \\ a_n = 0 \quad \forall \{n \in \mathbb{N} \setminus \{4, 6, 10\}\} \end{cases}$$

$$\therefore \boxed{u(x,t) = e^{-48t}\sin(4x) + 3e^{-108t}\sin(6x) - e^{-300t}\sin(10x)}$$

4. (Z 10.2 # 22)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 9\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = \sin(x) - \sin(2x) + \sin(3x) & \\ \dfrac{\partial u}{\partial t}(x,0) = 6\sin(3x) - 7\sin(5x) & \end{cases} \quad \Bigg\} \; 0 < x < \pi$$

step ①: Apply Sep'n of Vars

let $u(x,t) = F(x)G(t)$, then

$$F(x)G''(t) = 9F''(x)G(t) \iff /// $$

$$/// \iff \frac{G''(t)}{9G(t)} = \frac{F''(x)}{F(x)} = -\lambda$$

step ②: Form the ODE problems

$$\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} \iff \begin{cases} F(0)G(t) = 0 \\ F(\pi)G(t) = 0 \end{cases}$$

for nontrivial solutions we require $G(t) \neq 0$, &

$\therefore \; F(0) = F(\pi) = 0.$

$\therefore$ the ODE problems are

(A) $F''(x) + \lambda F(x) = 0, \quad F(0) = F(\pi) = 0$

(B) $G''(t) + 9\lambda G(t) = 0$

Step 3: Solve (A) & (B)

(A) This is the same BVP we solved in problem #3. $\therefore$ we know that the eigenvalues & eigenfunctions are

$$\begin{cases} \lambda_n = \rho^2 = n^2 \\ F_n(x) = \sin(nx) \end{cases}$$

(B) The corresponding solution to problem (B) is $G_n''(t) + 9n^2 G_n(t) = 0$ has characteristic equation $r^2 + 9n^2 = 0 \Leftrightarrow r = \pm 3ni$

$$\therefore G_n(t) = a_n \cos(3nt) + b_n \sin(3nt)$$

Step 4: Form the solution

$$u(x,t) = \sum_{n=1}^{\infty} F_n(x) G_n(t) = \sum_{n=1}^{\infty} \sin(nx) \left(a_n \cos(3nt) + b_n \sin(3nt)\right)$$

Step 5: Apply the initial conditions

$$\begin{cases} u(x,0) = \sum_{n=1}^{\infty} \sin(nx)(a_n) = \sin(x) - \sin(2x) + \sin(3x) \\ \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} 3n b_n \sin(nx) = 6\sin(3x) - 7\sin(5x) \end{cases}$$

$\therefore a_1 = 1,\ a_2 = -1,\ a_3 = 1$, all other $a_n = 0$

$3 \cdot 3 b_3 = 6 \Leftrightarrow b_3 = \frac{2}{3}$, $3 \cdot 5 b_5 = -7 \Leftrightarrow b_5 = -\frac{7}{15}$,

all other $b_n = 0$

$$\therefore \boxed{\begin{aligned} u(x,t) = {} & \sin(x)\cos(3t) - \sin(2x)\cos(6t) \\ & + \sin(3x)\left(\cos(9t) + \frac{2}{3}\sin(9t)\right) \\ & - \frac{7}{15}\sin(5x)\cos(15t) \end{aligned}}$$

Solve the IVP:

$$F''(x) + 4F'(x) + F(x) = 0$$

$$\begin{cases} F(0) = 0 \\ F'(0) = 2\sqrt{3} \end{cases}$$

and express the solution in terms of cosh or sinh:

$$r^2 + 4r + 1 = 0 \iff r = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$$

$$\therefore F(x) = \bar{c}_1 e^{(-2+\sqrt{3})x} + \bar{c}_2 e^{(-2-\sqrt{3})x}$$

or

$$= e^{-2x}\left(c_1 \cosh(\sqrt{3}x) + c_2 \sinh(\sqrt{3}x)\right)$$

apply ICs:

$$\begin{cases} F(0) = 0 \\ F'(0) = 2\sqrt{3} \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2\sqrt{3} = 2\sqrt{3} \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 = 2 \end{cases}$$

$$\therefore F(x) = 2e^{-2x} \sinh(\sqrt{3}x)$$

5. §10.2 #28

Given $\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

we apply $u(x,t) = X(x)T(t)$ & obtain

$$X(x)T''(t) + X(x)T'(t) + X(x)T(t) = \alpha^2 X''(x)T(t)$$

dividing through by $\alpha^2 X(x)T(t)$ we obtain

$$\frac{\cancel{X(x)}T''(t)}{\alpha^2 \cancel{X(x)}T(t)} + \frac{\cancel{X(x)}T'(t)}{\alpha^2 \cancel{X(x)}T(t)} + \frac{\cancel{X(x)}\cancel{T(t)}}{\alpha^2 \cancel{X(x)}\cancel{T(t)}} = \frac{\cancel{\alpha^2} X''(x)\cancel{T(t)}}{\cancel{\alpha^2} X(x)\cancel{T(t)}}$$

$$\frac{1}{\alpha^2}\frac{T''(t)}{T(t)} + \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} + \frac{1}{\alpha^2} = \frac{X''(x)}{X(x)}$$

This side is a fn of t only. This side is a fn of x only

∵ x & t are independent variables, we ∴ have to assume that the LHS & RHS ratios are constant. That is

$$\frac{1}{\alpha^2}\frac{T''(t)}{T(t)} + \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} + \frac{1}{\alpha^2} = \frac{X''(x)}{X(x)} = -\lambda$$

We thus arrive at the two ODEs:

$$\frac{X''(x)}{X(x)} = -\lambda \iff X''(x) + \lambda X(x) = 0$$

$$\frac{T''(t)}{\alpha^2 T(t)} + \frac{T'(t)}{\alpha^2 T(t)} + \frac{1}{\alpha^2} = -\lambda \iff (\text{III})$$

$$(\text{III}) \iff T''(t) + T'(t) + T(t) = -\lambda \alpha^2 T(t)$$

$$\iff T''(t) + T'(t) + (1 + \lambda \alpha^2) T(t) = 0$$

as required.

6. Consider the IVP

$$F''(x) + 4F'(x) + F(x) = 0 \; ; \; F(0) = 0 \, , \; F'(0) = 2\sqrt{3}$$

(This is the same sort of problem we solved in Math 225. The only difference here is that we are asked to express the solution in terms of cosh + sinh.)

Char eqn:

$$r^2 + 4r + 1 = 0 \iff r = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$$

$\therefore$ we can write the general solution as

$$F(x) = c_1 e^{(-2+\sqrt{3})x} + c_2 e^{(-2-\sqrt{3})x}$$

$$= e^{-2x}\left[c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}\right]$$

OR

$$F(x) = e^{-2x}\left[c_1 \cosh(\sqrt{3}x) + c_2 \sinh(\sqrt{3}x)\right]$$

using the second expression, we apply the ICs to solve for c_1 & c_2:

$$\begin{cases} F(0) = 0 \\ F'(0) = 2\sqrt{3} \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ \sqrt{3}\, c_2 = 2\sqrt{3} \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 2 \end{cases}$$

$$\therefore \boxed{F(x) = 2e^{-2x}\sinh(\sqrt{3}x)}$$

7. §10.3 #2

$$f(-x) = \sin^2(-x) = [\sin(-x)]^2 = [-\sin(x)]^2 = \sin^2(x) = f(x)$$

$\therefore$ $f(x)$ is <u>even</u>

8. § 10.3 # 6

$f(-x) = (-x)^{1/5} \cos((-x)^2) = -(x)^{1/5} \cos(x^2)$

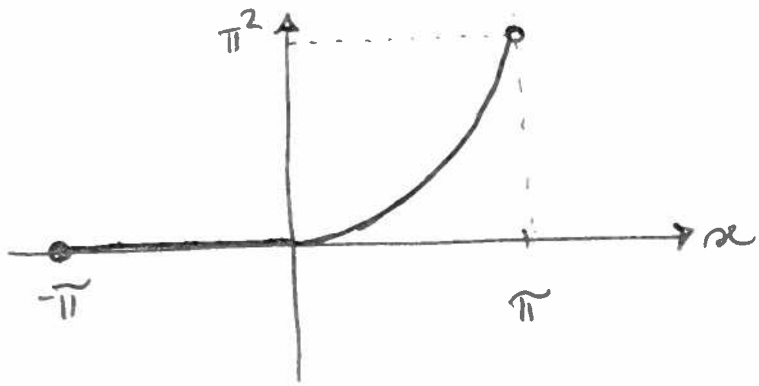
$= -f(x)$

$\therefore f(x)$ is odd.

(go to next p)

9. §10.3 #12

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$$



$L = \pi$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos\left(\frac{n\pi x}{\pi}\right)dx + \int_0^{\pi} x^2\cos\left(\frac{n\pi x}{\pi}\right)dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

let $u = x^2$ $\quad du = 2x\,dx$
$dv = \cos(nx)dx$ $\quad v = +\frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\frac{1}{n}x^2\sin(nx)\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx\right]$$

let
$u = x$ $\quad du = dx$
$dv = \sin(nx)dx$
$v = -\frac{1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[0 - \frac{2}{n}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right]\right]$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[-\frac{2}{n}\left(\left[\frac{-\pi\cos(n\pi)}{n} + 0\right] + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos(n\pi) + \cancel{\frac{1}{n^2}\sin(n\pi)}^{\,0} - 0\right]$$

$$= \frac{2}{n^2}(-1)^n$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2\sin\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx$$

let $u = x^2$ $\quad du = 2x\,dx$

$dv = \sin(nx)\,dx \quad v = -\frac{1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[-\frac{x^2}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{2x}{n}\cos(nx)\,dx\right]$$

let $u = x$ $\quad du = dx$

$dv = \cos(nx)\,dx \quad v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos(n\pi) + 0 + 2\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)\,dx\right]\right]$$

$$\therefore b_n = \frac{1}{\pi}\left[-\frac{\pi^2}{n}(-1)^n + 2\left(\cancel{\frac{\pi}{n}\sin(n\pi)}^{\,0} - 0 + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^2}\left(\cos(n\pi) - 1\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^2}\left((-1)^n - 1\right)\right]$$

$$= \frac{1}{\pi n}\left[\pi^2(-1)^{n+1} + \frac{2}{n}\left((-1)^n - 1\right)\right] = \frac{1}{n}\left(\pi(-1)^{n+1} + \frac{2}{n\pi}\left((-1)^n - 1\right)\right)$$

We also need to determine a_0:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\,\frac{x^3}{3}\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^3}{3}\right) = \frac{\pi^2}{3}$$

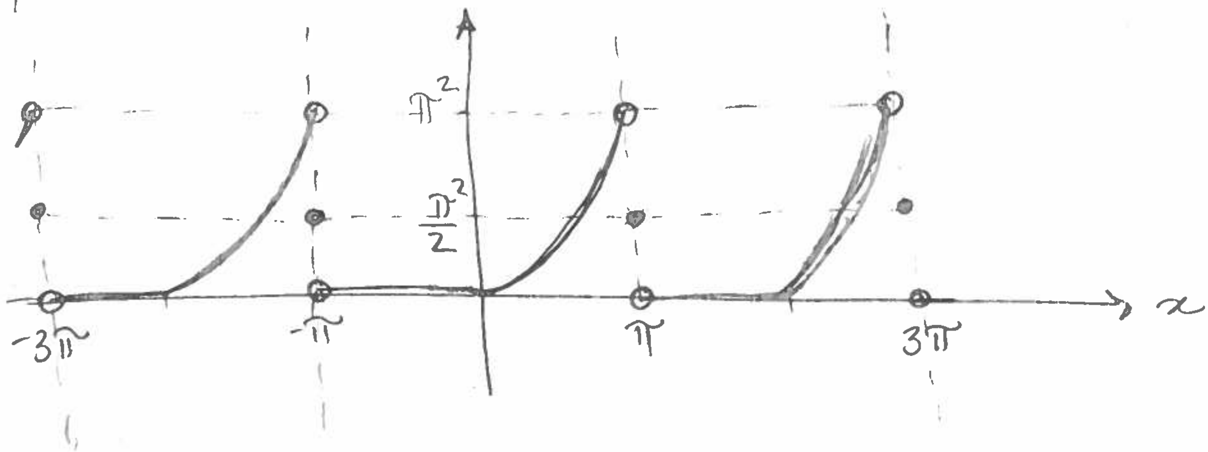
$\therefore$ the Fourier series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos(nx) + \frac{1}{n}\left(\pi(-1)^{n+1} + \frac{2}{n\pi}\left((-1)^n - 1\right)\right)\sin(nx)\right]$$

(plots are in a separate file)

10.7, 10.3 #20



The Fourier series converges to the function shown. Mathematically the Fourier series converges to the 2π-periodic function

$$g(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x^2 & 0 < x < \pi \\ \frac{\pi^2}{2} & x = \pm\pi \end{cases}$$