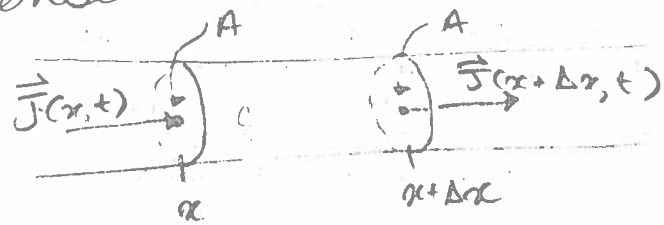
Assignment #3
Solutions

1. Conservation law:



$$\begin{bmatrix}\text{change}\\ \text{in amt}\\ \text{of stuff}\end{bmatrix} = \begin{bmatrix}\text{stuff}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{stuff}\\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{stuff}\\ \text{created}\end{bmatrix}$$

$$\frac{\partial}{\partial t}\left(c(x,t)A\Delta x\right) = \vec{J}(x,t)A - \vec{J}(x+\Delta x,t)A + \beta c(x) A\Delta x$$

$\Rightarrow \frac{1}{A\Delta x}$

$$\tfrac{1}{A\Delta x} \Rightarrow \frac{\partial}{\partial t}c(x,t) = \frac{\vec{J}(x,t) - J(x+\Delta x,t)}{\Delta x} + \beta c(x)$$

In the limit as $\Delta x \to 0$ we have

$$\frac{\partial c}{\partial t} = -\frac{\partial \vec{J}}{\partial x} + \beta c$$

Using Fick's law, $\vec{J}(x,t) = -D\frac{\partial c}{\partial x}$, we obtain

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + \beta c$$

2. §10.5 #1

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \qquad 0<x<1,\ t>0 \qquad \ldots\ldots (1)$$

$$u(0,t) = u(1,t) = 0 \qquad t>0 \qquad \ldots\ldots (2)$$

$$u(x,0) = (1-x)x^2 \qquad 0<x<1 \qquad \ldots\ldots (3)$$

Solution:

The PDE + BCs are homogeneous, so we start with separation of variables.

$\boxed{\text{Step 1}}$ $u(x,t) = F(x)G(t)$

(1) becomes:

$$FG' = 5F''G \Leftrightarrow \underbrace{\frac{G'}{5G}}_{\text{dep on } t} = \underbrace{\frac{F''}{F}}_{\text{dep. on } x} = \underset{\text{a constant}}{-\lambda}$$

(2) becomes

$$\begin{cases} F(0)G(t) = 0 \\ F(1)G(t) = 0 \end{cases}$$ for nontrivial solutions we require $F(0) = F(1) = 0$

(3) is not separable

$\boxed{\text{step 2}}$ We arrive at two ODE problems:

(A) BVP:

$$\begin{cases} F'' + \lambda F = 0 \\ F(0) = F(1) = 0 \end{cases}$$

(B) t-dependent

$$G' + \lambda G = 0$$

$\boxed{\text{step 3}}$

Solve (A)

char. eqn is $r^2 + \lambda = 0 \Leftrightarrow r^2 = -\lambda \Leftrightarrow r = \pm\sqrt{-\lambda}$

case 1: $\lambda < 0$, $\lambda = -\mu^2$

Then $F(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$.

When we apply the BCs we find

$$\begin{cases} F(0) = 0 \\ F(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

So only trivial solutions are possible in this case.

case 2: $\lambda = 0$

Then $F(x) = Ax + B$.

Applying the BCs:

$$\begin{cases} F(0) = 0 \\ F(1) = 0 \end{cases} \Leftrightarrow \begin{cases} B = 0 \\ A = 0 \end{cases}$$

So only trivial solns are possible in this case.

case 3: $\lambda > 0$, $\lambda = \mu^2$

Then $F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

Applying the BCs:

$$\begin{cases} F(0) = 0 \\ F(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 \sin(\mu) = 0 \end{cases}$$

For nontrivial sol'ns we require

$$\mu_n = n\pi \quad , \; n \in \mathbb{N} \qquad \text{(eigenvalues)}$$

and then

$$F_n(x) = \sin(n\pi x) \qquad \text{(eigenfunctions)}$$

Solve (B)

$$G' + 5\lambda G = 0 \Leftrightarrow G(t) = c_3 e^{-5\lambda t}$$

$$\text{or } G_n(t) = c_3 e^{-5(n\pi)^2 t} \quad , \; n \in \mathbb{N}$$

$\boxed{\text{step 4}}$ Form the formal solution:

$$u(x,t) = F(x)\, G(t)$$
$$= \sum_{n=1}^{\infty} b_n e^{-5(n\pi)^2 t} \sin(n\pi x)$$

$\boxed{\text{step 5}}$ Apply the IC (3):

$$u(x,0) = (1-x)x^2 \Leftrightarrow (1-x)x^2 = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

This is a Fourier sine series, and so

$$b_n = \frac{2}{1}\int_0^1 (1-x)x^2 \sin(n\pi x)\,dx$$

$$= 2\Big[\underbrace{\int_0^1 x^2 \sin(n\pi x)\,dx}_{\mathrm{I}} - \underbrace{\int_0^1 x^3 \sin(n\pi x)\,dx}_{\mathrm{II}}\Big]$$

Solving integral I:

$$\mathrm{I} = \int_0^1 x^2 \sin(n\pi x)\,dx$$

let $u = x^2$ $\quad du = 2x\,dx$

$dv = \sin(n\pi x)\,dx$ $\quad v = -\frac{\cos(n\pi x)}{n\pi}$

$$= -\frac{x^2\cos(n\pi x)}{n\pi}\Big|_0^1 + \int_0^1 \frac{2x}{n\pi}\cos(n\pi x)\,dx$$

let $u = x$ $\quad du = dx$

$dv = \cos(n\pi x)\,dx$ $\quad v = \frac{\sin(n\pi x)}{n\pi}$

$$= -\frac{\cos(n\pi)}{n\pi} + 0 + \frac{2}{n\pi}\left[\frac{x\sin(n\pi x)}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx\right]$$

$$= -\frac{\cos(n\pi)}{n\pi} + \frac{2}{n\pi}\left[\cancel{\frac{\sin(n\pi)}{n\pi}}^{\,0} - 0 + \frac{\cos(n\pi)}{(n\pi)^2}\Big|_0^1\right]$$

$$= -\frac{\cos(n\pi)}{n\pi} + \frac{2}{(n\pi)^3}\left[\cos(n\pi) - 1\right]$$

checked ✓

OR $I = -\frac{(-1)^n}{n\pi} + \frac{2}{(n\pi)^3}\left[(-1)^n - 1\right]$

Solving integral II:

$$II = \int_0^1 x^3 \sin(n\pi x)\,dx$$

let $u = x^3$ $du = 3x^2$

$dv = \sin(n\pi x)\,dx$ $v = -\frac{\cos(n\pi x)}{n\pi}$

$$= -\frac{x^3\cos(n\pi x)}{n\pi}\Bigg|_0^1 + \int_0^1 \frac{3}{n\pi} x^2 \cos(n\pi x)\,dx$$

let $u = x^2$ $du = 2x\,dx$

$dv = \cos(n\pi x)\,dx$ $v = \frac{\sin(n\pi x)}{n\pi}$

$$= -\frac{\cos(n\pi)}{n\pi} + 0 + \frac{3}{n\pi}\left[\frac{x^2\sin(n\pi x)}{n\pi}\Bigg|_0^1 - \int_0^1 \frac{2x\sin(n\pi x)}{n\pi}\,dx\right]$$

$$= -\frac{\cos(n\pi)}{n\pi} - \frac{6}{(n\pi)^2}\int_0^1 x\sin(n\pi x)\,dx$$

let $u = x$ $du = dx$

$dv = \sin(n\pi x)\,dx$ $v = -\frac{\cos(n\pi x)}{n\pi}$

$$= -\frac{\cos(n\pi)}{n\pi} - \frac{6}{(n\pi)^2}\left[-\frac{x\cos(n\pi x)}{n\pi}\Bigg|_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx\right]$$

$$\text{II} = -\frac{\cos(n\pi)}{n\pi} + \frac{6}{(n\pi)^3}\cos(n\pi) - \frac{6}{(n\pi)^3}\int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{\cos(n\pi)}{(n\pi)^3}\left[6 - (n\pi)^2\right] - \frac{6}{(n\pi)^3}\,\frac{\sin(n\pi x)}{n\pi}\Bigg|_0^1$$

$$= \frac{\left[6-(n\pi)^2\right]}{(n\pi)^3}\cos(n\pi) \quad \text{OR} \quad \frac{\left[6-(n\pi)^3\right]}{(n\pi)^3}(-1)^n$$

Thus,

$$b_n = 2\left[\text{I} - \text{II}\right]$$

$$= 2\left(\frac{\left[2-(n\pi)^2\right]}{(n\pi)^3}\cos(n\pi) - \frac{2}{(n\pi)^3} - \frac{\left[6-(n\pi)^2\right]\cos(n\pi)}{(n\pi)^3}\right)$$

$$= \frac{2}{(n\pi)^3}\left(\left[\left(2-(n\pi)^2\right) - \left(6-(n\pi)^2\right)\right]\cos(n\pi) - 2\right)$$

$$= \frac{2}{(n\pi)^3}\left(\left[2-(n\pi)^2 - 6 + (n\pi)^2\right]\cos(n\pi) - 2\right)$$

$$= \frac{2}{(n\pi)^3}\left(-4\cos(n\pi) - 2\right)$$

$$= \frac{-4}{(n\pi)^3}\left(2\cos(n\pi) + 1\right) \quad \text{OR} \quad \frac{-4}{(n\pi)^3}\left(2(-1)^n + 1\right)$$

Combining this result with the expression in $\boxed{\text{step 4}}$

we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-4}{(n\pi)^3}\left[2\cos(n\pi)+1\right] e^{-(n\pi)^2 t} \sin(n\pi x)$$

or

$$= \sum_{n=1}^{\infty} \frac{2}{(n\pi)^3}\left[2(-1)^n+1\right] e^{-(n\pi)^2 t} \sin(n\pi x)$$

#3. (§ 10.5 #7)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{2\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 & \cdots\cdots (4) \\ u(0,t) = 5, \quad u(\pi,t) = 10, \quad t > 0 & \cdots\cdots (5) \\ u(x,0) = \sin(3x) - \sin(5x), \quad 0 < x < \pi & \cdots\cdots (6) \end{cases}$$

Sol'n

Since the BCs are nonhomogeneous, we assume

$$u(x,t) = v(x) + w(x,t) \quad \cdots\cdots (7)$$

where $\lim_{t\to\infty}(w(x,t)) = 0$

Plugging (7) into (4) we obtain

$$w_t = 2(v'' + w_{xx}) \quad \cdots\cdots (8)$$

Letting $t \to \infty$, (8) becomes

$$v'' = 0 \Leftrightarrow v(x) = Ax + B \quad \cdots\cdots (9)$$

Applying (5) we have

$$\begin{cases} v(0) = 5 \\ v(\pi) = 10 \end{cases} \Leftrightarrow \begin{cases} B = 5 \\ A\pi + 5 = 10 \end{cases} \Leftrightarrow \begin{cases} B = 5 \\ A = \dfrac{5}{\pi} \end{cases}$$

$$\therefore v(x) = \frac{5}{\pi}x + 5 \quad \cdots\cdots (10)$$

Now we plug (10) into (8) and (5) and obtain

$$\begin{cases} w_t = 2w_{xx} & (11) \\ w(0,t) = w(\pi,t) = 0 & (12) \end{cases}$$

To solve (11) with (12) we assume

step1

$$w(x,t) = F(x)G(t). \quad (13)$$

Plugging (13) into (11) we obtain

$$FG' = 2F''G \Leftrightarrow \underbrace{\frac{G'}{2G}}_{t\text{-dependent}} = \underbrace{\frac{F''}{F}}_{x\text{-dependent}} = -\lambda \quad (14)$$

↑ constant

step2

Equation (14) with (13) & (12) corresponds to

(A) $\begin{cases} F'' + \lambda F = 0 \\ F(0) = F(\pi) = 0 \end{cases}$ (B) $G' + 2\lambda G = 0$

step3

We first solve problem (A). The characteristic equation for $F(x)$ is

$$r^2 + \lambda = 0 \Leftrightarrow r = \pm\sqrt{-\lambda}$$

<u>case 1</u>: <u>$\lambda < 0$</u>, $\lambda = -\mu^2$

Then $F(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$. Applying the BCs we obtain

$$F(0) = 0 \Leftrightarrow c_1 = 0, \quad F(\pi) = 0 \Leftrightarrow c_2 = 0$$

$\therefore$ only trivial solutions are possible in this case.

case 2: $\lambda = 0$

Then $F(x) = c_1 + c_2 x$. Applying the BCs we obtain

$$F(0) = 0 \Leftrightarrow c_1 = 0, \quad F(\pi) = 0 \Leftrightarrow c_2 = 0$$

$\therefore$ only trivial solutions are possible in this case.

case 3: $\lambda > 0$, $\lambda = \mu^2$

Then $F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. Applying the BCs we obtain

$$F(0) = 0 \Leftrightarrow c_1 = 0$$

$$F(\pi) = 0 \Leftrightarrow c_2 \sin(\mu \pi) = 0$$

Nontrivial solutions are possible if

$$\mu = n, \quad n \in \mathbb{N} \quad (\text{eigenvalues}) \quad \ldots \quad (15)$$

The corresponding eigenfunctions are

$$F_n(x) = \sin(nx), \; n \in \mathbb{N} \quad \ldots \quad (16)$$

Using (15) we can now solve problem (B).

$$G_n' + 2n^2 G_n = 0 \Leftrightarrow G_n = c_n e^{-2n^2 t} \quad \ldots \quad (17)$$

$\boxed{\text{step 4}}$

Plugging (16) & (17) into (13) we have

$$W(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-2n^2 t} \quad \ldots \quad (18)$$

$\boxed{\text{Step 5}}$ Applying (6) to (10) and (18) we have

$$u(x,0) = \sin(3x) - \sin(5x) \Leftrightarrow //$$

$$// \Leftrightarrow v(x) + w(x,0) = \sin(3x) - \sin(5x)$$

$$\Leftrightarrow \frac{5}{\pi}x + 5 + \sum_{n=1}^{\infty} c_n \sin(nx) = \sin(3x) - \sin(5x)$$

$$\Leftrightarrow \sum_{n=1}^{\infty} c_n \sin(nx) = \underbrace{\sin(3x) - \sin(5x) - \frac{5x}{\pi} - 5}_{f(x)}$$

Thus, we have

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left[\sin(3x) - \sin(5x) - \frac{5x}{\pi} - 5\right]\sin(nx)\,dx$$

$$= \frac{2}{\pi}\Bigg[\underbrace{\int_0^{\pi}\sin(3x)\sin(nx)\,dx}_{\text{I}} - \underbrace{\int_0^{\pi}\sin(5x)\sin(nx)\,dx}_{\text{II}}$$

$$- \frac{5}{\pi}\underbrace{\int_0^{\pi} x\sin(nx)\,dx}_{\text{III}} - 5\underbrace{\int_0^{\pi}\sin(nx)\,dx}_{\text{IV}}$$

case 1: $n = 3$

$$I = \int_0^{\pi} \sin^2(3x)\,dx = \int_0^{\pi} \frac{1-\cos(6x)}{2}\,dx$$

$$= \frac{1}{2}x \Big|_0^{\pi} - \frac{1}{12}\sin(6x)\Big|_0^{\pi}$$

$$= \frac{\pi}{2}.$$

$$II = 0 \quad \text{(by orthogonality)}$$

$$III = -\frac{5}{\pi}\int_0^{\pi} x\sin(3x)\,dx$$

let $u = x$

$du = dx$

$dv = \sin(3x)\,dx$

$v = -\frac{1}{3}\cos(3x)$

$$= -\frac{5}{\pi}\left[-\frac{x}{3}\cos(3x)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{3}\cos(3x)\,dx\right]$$

$$= -\frac{5}{\pi}\left[\frac{\pi}{3} + \frac{1}{9}\sin(3x)\Big|_0^{\pi}\right]$$

$$= -\frac{5}{3}$$

$$IV = -5\int_0^{\pi}\sin(3x)\,dx = +\frac{5}{3}\cos(3x)\Big|_0^{\pi} = +\frac{5}{3}(-1-1)$$

$$= -\frac{10}{3}.$$

$$\therefore\ c_3 = \frac{2}{\pi}\left[\frac{\pi}{2} - \frac{5}{3} - \frac{10}{3}\right] = \frac{2}{\pi}\left[\frac{\pi}{2} - \frac{15}{3}\right] = \left(1 - \frac{30}{3\pi}\right) = 1 - \frac{10}{\pi}$$

case 2 : n = 5

$\mathrm{I} = 0$ (by orthogonality)

$$\mathrm{II} = -\int_0^\pi \sin^2(5x)\,dx = -\int_0^\pi \frac{1-\cos(10x)}{2}\,dx$$

$$= -\left[\frac{1}{2}x\Big|_0^\pi + 0\right] = -\frac{\pi}{2}$$

$$\mathrm{III} = -\frac{5}{\pi}\int_0^\pi x\sin(5x)\,dx = -\frac{5}{\pi}\left[-\frac{x}{5}\cos(5x)\Big|_0^\pi + \int_0^\pi \frac{1}{5}\cos(5x)\,dx\right]$$

$$= -\frac{5}{\pi}\left[\frac{\pi}{5} + \frac{1}{25}\sin(5x)\Big|_0^\pi\right] = -1$$

$$\mathrm{IV} = -5\int_0^\pi \sin(5x)\,dx = +\frac{5}{5}\cos(5x)\Big|_0^\pi = -1-1 = -2$$

$$\therefore\ c_5 = \frac{2}{\pi}\left[0 - \frac{\pi}{2} - 1 - 2\right] = \frac{2}{\pi}\left[-\frac{\pi}{2} - 3\right] = \left(-1 - \frac{6}{\pi}\right)$$

case 3: $n \neq 3$ + $n \neq 5$

$\underline{I} = 0$ and $\underline{II} = 0$ (by orthogonality)

$$\underline{III} = \frac{-5}{\pi}\int_0^\pi x\sin(nx)\,dx = -\frac{5}{\pi}\left[-\frac{x}{n}\cos(nx)\Big|_0^\pi + \int_0^\pi \frac{1}{n}\cos(nx)\,dx\right]$$

$$= -\frac{5}{\pi}\left[\frac{-\pi\cos(n\pi)}{n} + \frac{1}{n^2}\sin(nx)\Big|_0^\pi\right]$$

$$= \frac{5\cos(n\pi)}{n} = \frac{5(-1)^n}{n}$$

$$\underline{IV} = -5\int_0^\pi \sin(nx)\,dx = \frac{5}{n}\cos(nx)\Big|_0^\pi$$

$$= \frac{5}{n}\left(\cos(n\pi) - 1\right) = \frac{5}{n}\left((-1)^n - 1\right)$$

$$\therefore\ c_n = \frac{2}{\pi}\left[\frac{5(-1)^n}{n} + \frac{5}{n}\left((-1)^n - 1\right)\right] = \frac{2}{\pi}\left[\frac{10(-1)^n}{n} - \frac{5}{n}\right]$$

$$= \frac{10}{n\pi}\left(2(-1)^n - 1\right)$$

Our final solution is thus

$$u(x,t) = v(x) + w(x,t)$$

$$= \frac{5}{\pi}x + 5 + \sum_{n=1}^{\infty} c_n \sin(nx) e^{-2n^2 t}$$

where

$$\begin{cases} c_3 = 1 - \frac{10}{\pi} \\ c_5 = -1 - \frac{6}{\pi} \\ c_n = \frac{10}{n\pi}\left[2(-1)^n - 1\right], \quad n \in \mathbb{N} \setminus \{3,5\} \end{cases}$$

Could also be simply written:

$$u(x,t) = \frac{5}{\pi}x + 5 + \left[\sin(3x) - \sin(5x) + \sum_{n=1}^{\infty} \frac{10}{n\pi}\left[2(-1)^n - 1\right] \sin(nx)\right] e^{-2n^2 t}$$

#4. (Z 10.5 #18)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \quad \text{(19)} \\ \dfrac{\partial u}{\partial x}(0,y,t) = \dfrac{\partial u}{\partial x}(\pi,y,t) = 0, & 0 < y < \pi,\ t > 0 \quad \text{(20)} \\ u(x,0,t) = u(x,\pi,t) = 0, & 0 < x < \pi,\ t > 0 \quad \text{(21)} \\ u(x,y,0) = x\sin(y), & 0 < x < \pi,\ 0 < y < \pi \quad \text{(22)} \end{cases}$$

Assume

$\boxed{\text{step 1}}$

$$u(x,y,t) = F(x)\,G(y)\,H(t) \quad \text{(23)}$$

Then (19) becomes

$$FGH' = F''GH + FG''H \iff {}^{1}/$$

$$/_1 \iff \underbrace{\frac{H'}{H}}_{t\text{-dep}} = \underbrace{\frac{F''}{F} + \frac{G''}{G}}_{(x,y)\text{ dep}} = \underset{\text{cst}}{-\lambda}$$

This gives us

$$H' + \lambda H = 0 \quad \text{(24)}$$

$$\frac{F''}{F} + \frac{G''}{G} = -\lambda \iff \underbrace{\frac{F''}{F}}_{x\text{-dep}} = \underbrace{-\frac{G''}{G} - \lambda}_{y\text{-dep}} = \underset{\text{cst}}{-\mu} \quad \text{(25)}$$

(24) + (25) give us linear ODE problems, and separating BCs (20) + (21) we arrive at

step 2.

(A) $\begin{cases} F'' - \gamma F = 0 \\ F'(0) = F'(\pi) = 0 \end{cases}$ (B) $\begin{cases} G'' + (\lambda - \gamma) G = 0 \\ G(0) = G(\pi) = 0 \end{cases}$

(C) $H' + \lambda H = 0$

step 3

We solve each of these problems in turn. First, we solve problem (A). The characteristic eqn for $F(x)$ is

$$r^2 + \gamma = 0 \iff r^2 = -\gamma \iff r = \pm\sqrt{-\gamma}$$

case 1: $\gamma < 0$, $\gamma = \mu^2$

$F(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$

$F'(x) = c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x)$

Applying the BCs we find

$$\begin{cases} F'(0) = 0 \\ F'(\pi) = 0 \end{cases} \iff \begin{cases} c_2 = 0 \\ c_1 \mu \sinh(\mu\pi) = 0 \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 = 0 \quad (\because \mu \neq 0) \end{cases}$$

$\therefore$ only trivial sol'ns are possible in this case

case 2: $\gamma = 0$,

$F(x) = Ax + B$, $F'(x) = A$

Applying the BCs we find $F'(0) = F'(\pi) = 0 \iff A = 0$

$\therefore$ $F_0(x) = B$ is a possible solution

<u>case 3: $\gamma > 0$</u>, $\gamma = \mu^2$

$F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$F'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$

Applying the BCs we find

$$\begin{cases} F'(0) = 0 \\ F'(\pi) = 0 \end{cases} \Leftrightarrow \begin{cases} c_2 = 0 \\ -c_1 \mu \sin(\mu\pi) = 0 \end{cases}$$

For nontrivial solutions, we require $\mu\pi = n\pi \Leftrightarrow \boxed{\mu = n}$, $n \in \mathbb{N}$

The corresponding eigenfunctions are

$$F_n(x) = \cos(nx) \qquad \ldots\ldots\ldots (26)$$

Now we solve problem (B). If we let $\beta = \lambda - \gamma$, then problem B is written

$$\begin{cases} G'' + \beta G = 0 \\ G(0) = G(\pi) = 0 \end{cases}$$

We solved this BVP in #3 (problem (A) on p(10)), so we can write down the solution right away:

$$\begin{cases} \beta = m^2, \ m \in \mathbb{N} \\ G_m(y) = \sin(my) \end{cases} \qquad \ldots\ldots (27)$$

Now we solve problem (C) Note that

$$\lambda - \gamma = \beta \Leftrightarrow \lambda - n^2 = m^2 \Leftrightarrow \lambda = n^2 + m^2$$

Thus, problem (C) is written

$$H'_{n,m} + (n^2+m^2) H_{n,m} = 0 \iff H_{n,m}(t) = a_{n,m} e^{-(n^2+m^2)t} \quad \ldots\ldots (28)$$

$\boxed{\text{step 4}}$ Combining (26), (27), and (28) we have

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \cos(nx) \sin(my) e^{-(n^2+m^2)t} \quad \ldots\ldots (29)$$

$\boxed{\text{step 5}}$ Apply the IC (22):

$$u(x,y,0) = x \sin(y) \iff ①$$

$$① \iff x \sin(y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} \cos(nx) \sin(my)$$

$\boxed{\text{step 6}}$ We determine the Fourier coefficients using equations (54) and (55) from the text:

$$a_{0m} = \frac{2}{\pi^2} \int_0^{\pi} \int_0^{\pi} x \sin(y) \sin(my)\, dy\, dx \quad \ldots\ldots\ldots\ldots (30)$$

case 1: $m \neq 1$; $a_{0m} = 0$

case 2: $m = 1$

$$a_{0,1} = \frac{2}{\pi^2} \int_0^{\pi} \int_0^{\pi} x \sin^2(y)\, dy\, dx$$

using $\sin^2(y) = \dfrac{1-\cos(2y)}{2}$ we have

$$a_{01} = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi \frac{x}{2}\left(1-\cos(2y)\right)dy\,dx$$

$$= \frac{2}{\pi^2}\int_0^\pi \frac{x}{2}\left[y - \frac{\sin(2y)}{2}\right]\Bigg|_0^\pi dx$$

$$= \frac{2}{\pi^2}\left[\frac{x^2}{4}\Bigg|_0^\pi (\pi)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}} \quad (3)$$

and

$$a_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi x\sin(y)\cos(mx)\sin(ny)\,dy\,dx$$

$$= \frac{4}{\pi^2}\int_0^\pi x\cos(mx)\left[\int_0^\pi \sin(y)\sin(ny)\,dy\right]dx$$

Case 1: $m \neq 1$, $a_{mn} = 0$

case 1: $m = 1$

$$a_{n1} = \frac{4}{\pi^2}\int_0^\pi x\cos(nx)\left[\int_0^\pi \sin^2(y)\,dy\right]dx$$

$$= \frac{4}{\pi^2}\int_0^\pi x\cos(nx)\left[\frac{y - \sin(2y)}{2}\right]_0^\pi dx = I$$

$$I = \frac{2}{\pi^2}\int_0^{\pi} \pi x \cos(nx)\, dx$$

let $u = x$ $\quad du = dx$

$dv = \cos(nx)\,dx \quad v = \frac{\sin(nx)}{n}$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[\;0 + \left[\frac{\cos(nx)}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] \quad \left(\text{or } \frac{4}{\pi}\,\frac{\cos(n\pi)}{n^2}\right) \quad \ldots\ldots (32)$$

Now we can write the final solution:

$$\boxed{u(x,y,t) = \frac{\pi}{2}\sin(y)\,e^{-t} + \sum_{n=1}^{\infty}\frac{2[(-1)^n-1]}{\pi n^2}\cos(nx)\sin(y)\,e^{-(n^2+1)t}}$$

OR, $\because$ even coefficients are zero,

$$u(x,y,t) = \frac{\pi}{2}\sin(y)e^{-t} + \sum_{m=1}^{\infty}\frac{-4}{\pi(2m-1)^2}\cos\left((2m-1)x\right)\sin(y)\,e^{-\left((2m-1)^2+1\right)t}$$

#5. Show that the set

$$\left\{ \sin\left(\frac{(2n-1)\pi x}{2a}\right) \right\}_{n=1}^{\infty}$$

is orthogonal on $[0,a]$ with respect to the weight function $w(x) = 1$.

We need to show that

$$I = \int_0^a \sin\left(\frac{(2n-1)\pi x}{2a}\right) \sin\left(\frac{(2m-1)\pi x}{2a}\right) dx = 0 \quad \forall\, m \neq n$$

We know that

$$\sin(A)\sin(B) = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$\therefore \quad I = \frac{1}{2}\int_0^a \left[\cos\left(\frac{\pi}{2a}\left[(2n-1)x - (2m-1)x\right]\right) - \cos\left(\frac{\pi}{2a}\left[(2n-1)x + (2m-1)x\right]\right)\right] dx$$

$$I = \frac{1}{2}\int_0^a \left[\cos\left(\frac{\pi}{\cancel{2}a}\cancel{2}(n-m)x\right) - \cos\left(\frac{\pi}{\cancel{2}a}\cancel{2}\left[(n+m)-1\right]x\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{a}{\pi(n-m)} \sin\left(\frac{\pi}{a}(n-m)x\right) - \frac{a}{\pi(n+m-1)} \sin\left(\frac{\pi}{a}\left[(n+m)-1\right]x\right)\right]_0^a$$

$$= \frac{a}{2\pi}\left[\frac{1}{(n-m)} \sin\cancelto{0}{\left((n-m)\pi\right)} - \frac{1}{(n+m-1)} \sin\cancelto{0}{\left((n+m-1)\pi\right)}\right]$$

$= 0$, as required.