Math 319

Assignment #4 - Solutions

1. $$\frac{d^2 u_n(t)}{dt^2} + \left(\frac{n\pi\alpha}{L}\right)^2 u_n(t) = h_n(t) \quad \ldots\ldots\ldots \quad (1)$$

The homogeneous equation ($h_n(t)=0$) has solutions

$$u_{n_h}(t) = c_1 \cos\left(\frac{n\pi\alpha}{L}t\right) + c_2 \sin\left(\frac{n\pi\alpha}{L}t\right) \quad \ldots\ldots \quad (2)$$

For the nonhomogeneous equation, we thus look for a particular solution of the form

$$u_{n_p}(t) = v_1(t)\cos\left(\frac{n\pi\alpha}{L}t\right) + v_2(t)\sin\left(\frac{n\pi\alpha}{L}t\right). \ldots \quad (3)$$

In order for (3) to be a solution of (1), $v_1(t)$ and $v_2(t)$ must satisfy

$$\begin{cases} v_1' \cos\left(\frac{n\pi\alpha}{L}t\right) + v_2' \sin\left(\frac{n\pi\alpha}{L}t\right) = 0 \\ -v_1'\left(\frac{n\pi\alpha}{L}\right)\sin\left(\frac{n\pi\alpha}{L}t\right) + v_2'\left(\frac{n\pi\alpha}{L}\right)\cos\left(\frac{n\pi\alpha}{L}t\right) = h_n(t) \end{cases} \quad \Leftrightarrow (*)$$

Let $\frac{n\pi\alpha}{L} = \mu$. Then rewrite (*):

$$\begin{cases} v_1' \cos(\mu t) + v_2' \sin(\mu t) = 0 & \cdots\cdots (4) \\ -v_1' \sin(\mu t) + v_2' \cos(\mu t) = \dfrac{h_n(t)}{\mu} & \cdots\cdots (5) \end{cases}$$

If we take $[\sin(\mu t) \times (4) + \cos(\mu t) \times (5)]$ and $[\cos(\mu t) \times (4) - \sin(\mu t) \times (5)]$ we arrive at

$$\begin{cases} v_2' \left(\sin^2(\mu t) + \cos^2(\mu t)\right) = \dfrac{h_n(t)}{\mu} \cos(\mu t) \\ v_1' \left(\cos^2(\mu t) + \sin^2(\mu t)\right) = -\dfrac{h_n(t)}{\mu} \sin(\mu t) \end{cases} \Leftrightarrow //$$

$$// \Leftrightarrow \begin{cases} v_2' = \dfrac{h_n(t)}{\mu} \cos(\mu t) \\ v_1' = -\dfrac{h_n(t)}{\mu} \sin(\mu t) \end{cases} \Leftrightarrow \begin{cases} v_2 = \dfrac{1}{\mu} \displaystyle\int h_n(t) \cos(\mu t)\, dt \\ v_1 = \dfrac{-1}{\mu} \displaystyle\int h_n(t) \sin(\mu t)\, dt \end{cases}$$

$$\therefore\ u_{n_p}(t) = v_2 \sin(\mu t) + v_1 \cos(\mu t)$$

$$= \frac{1}{\mu} \int h_n(s) \cos(\mu s) \sin(\mu t)\, ds - \frac{1}{\mu} \int h_n(s) \sin(\mu s) \cos(\mu t$$

$$\therefore u_{n_p}(t) = \frac{1}{\mu}\int h_n(s)\left[\cos(\mu s)\sin(\mu t) - \sin(\mu s)\cos(\mu t)\right]ds$$

$$= \frac{1}{\mu}\int h_n(s)\left[\sin(\mu t - \mu s)\right]ds$$

$$= \frac{1}{\mu}\int h_n(s)\sin(\mu(t-s))\,ds$$

Finally,

$$u_n(t) = u_{n_h}(t) + u_{n_p}(t)$$

$$\boxed{u_n(t) = c_1\cos\left(\frac{n\pi\alpha}{L}t\right) + c_2\sin\left(\frac{n\pi\alpha}{L}t\right) + \frac{1}{\mu}\int h_n(s)\sin(\mu(t-s))\,ds}$$

2. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1,\ t > 0 \quad \ldots\ldots (6) \\ u(0,t) = u(1,t) = 0 & t > 0 \quad \ldots\ldots (7) \\ u(x,0) = \sin(3\pi x) - 5\sin(8\pi x) + \dfrac{10}{3}\sin(12\pi x) & 0 < x < 1 \quad \ldots (8) \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & 0 < x < 1 \quad \ldots\ldots (9) \end{cases}$$

a) Assuming $u(x,t) = F(x)G(t)$, equation (6) becomes

$$\frac{G''}{G} = \frac{F''}{F} = -\lambda, \text{ a constant.} \quad \ldots\ldots (10)$$

Using (10) and separation of variables on (7) & (9) we arrive at two ODE problems

(A) $\begin{cases} F'' + \lambda F = 0 \\ F(0) = F(1) = 0 \end{cases}$ (B) $\begin{cases} G'' + \lambda G = 0 \\ G'(0) = 0 \end{cases}$

Solving Problem (A)

This is a BVP that only admits nontrivial solutions if $\lambda > 0$. We set $\lambda = \mu^2$ and obtain

$$\mu = n\pi, \quad F_n(x) = \sin(n\pi x), \quad n \in \mathbb{N} \quad (11)$$

Solving Problem (13)

Since λ is known, we have

$$G_n(t) = a_n \cos(n\pi t), \quad n \in \mathbb{N} \quad \ldots \quad (12)$$

Thus, we arrive at the formal solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi t), \quad n \in \mathbb{N} \quad \ldots \quad (13)$$

We now apply the condition (8):

$$u(x,0) = \sin(3\pi x) - 5\sin(8\pi x) + \frac{10}{3}\sin(12\pi x) \iff ///$$

$$/// \iff \sum_{n=1}^{\infty} a_n \sin(n\pi x) = \sin(3\pi x) - 5\sin(8\pi x) + \frac{10}{3}\sin(12\pi x) \quad \ldots \quad (14)$$

$\therefore \; a_3 = 1, \; a_8 = -5 \;\; \& \;\; a_{12} = \frac{10}{3}$. Plugging these values into (13) we obtain

$$\boxed{u(x,t) = \sin(3\pi x)\cos(3\pi t) - 5\sin(8\pi x)\cos(8\pi t) + \frac{10}{3}\sin(12\pi x)\cos(12\pi t)}$$

b) To obtain the Fourier coefficients we used the fact that the functions $\sin(n\pi x)$, $n \in \mathbb{N}$, are linearly independent, which allowed us to simply match the coefficients a_n on the LHS of (14) with the appropriate coefficients on the RHS of (14).

3. §10.6 #2

$$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \ t > 0 \quad \ldots\ldots (15)$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0 \quad \ldots\ldots (16)$$

$$u(x,0) = \sin^2(x), \quad 0 < x < \pi \quad \ldots\ldots (17)$$

$$\frac{\partial u}{\partial t}(x,0) = 1 - \cos(x), \quad 0 < x < \pi \quad \ldots\ldots (18)$$

We see that $\alpha = 4$ and $L = \pi$. Thus, a formal solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(4nt) + b_n\sin(4nt)\right]\sin(nx) \quad (1\text{c}$$

To find a_n & b_n, we use (17) & (18). Applying these two conditions, we find

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = \sin^2(x), \quad \ldots \ldots \quad (20)$$

and

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} 4 n b_n \sin(nx) = 1 - \cos(x). \quad \ldots \quad (21)$$

The series in (20) & (21) are Fourier sine series & so we can compute the coefficients. We first solve (20):

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin^2(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos(2x)}{2}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\Bigg[\underbrace{\int_0^{\pi}\sin(nx)\,dx}_{I_1} - \underbrace{\int_0^{\pi}\cos(2x)\sin(nx)\,dx}_{I_2}\Bigg]$$

$$I_1 = \left[-\frac{\cos(nx)}{n}\right]\Bigg|_0^{\pi} = \left[-\frac{\cos(n\pi)}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{n}\left[-(-1)^n + 1\right]$$

$$\int_0^\pi \cos(mx)\sin(nx)\,dx = I_2 \quad , \quad \underline{\underline{n \neq m}}$$

let $u = \sin(nx)$ $\qquad du = n\cos(nx)\,dx$

$dv = \cos(mx)\,dx$ $\qquad v = \frac{1}{m}\sin(mx)$

$$I_2 = \overbrace{\frac{1}{m}\sin(mx)\sin(nx)\Big|_0^\pi}^{\to 0} - \int_0^\pi \frac{n}{m}\sin(mx)\cos(nx)\,dx$$

$$= -\frac{n}{m}\int_0^\pi \sin(mx)\cos(nx)\,dx$$

let $u = \cos(nx)$ $\qquad du = -n\sin(nx)\,dx$

$dv = \sin(mx)\,dx$ $\qquad v = -\frac{1}{m}\cos(mx)$

$$= -\frac{n}{m}\left[-\frac{1}{m}\cos(mx)\cos(nx)\Big|_0^\pi - \int_0^\pi \frac{n}{m}\cos(mx)\sin(nx)\,dx\right]$$

$$= \frac{n}{m^2}\cos(mx)\cos(nx)\Big|_0^\pi + \frac{n^2}{m^2}\underbrace{\int_0^\pi \cos(mx)\sin(nx)\,dx}_{I_2}$$

$$\therefore I_2\left(1 - \frac{n^2}{m^2}\right) = \frac{n}{m^2}\left[(-1)^{m+n} - 1\right] \therefore \frac{n}{m^2-n^2}\left[(-1)^{n+m} - 1\right] = I_2$$

If $n = m$ then

$$I_2 = \int_0^{\pi} \cos(mx)\sin(mx)\,dx$$

$$= \int_0^{\pi} \frac{1}{2}\sin(2mx)\,dx$$

$$= \frac{1}{2}\,\frac{(-1)}{2m}\cos(2mx)\Big|_0^{\pi}$$

$$= -\frac{1}{4m}\left[\cos(2m\pi) - \cos(0)\right]$$

$$= -\frac{1}{4m}\left[(-1)^{2m} - 1\right]$$

$$= -\frac{1}{4m}\left[1 - 1\right]$$

$$= 0$$

$$\therefore\ a_2 = \frac{1}{n}\left[-(-1)^2 + 1\right] = 0 \quad \ldots\ldots\ldots \quad (21a)$$

For I_2, $m=2$, $\therefore$ $I_2 = \dfrac{n}{4-n^2}\left[(-1)^n - 1\right]$

and so
$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\left(1-(-1)^n\right) + \frac{n}{4-n^2}\left(1-(-1)^n\right)\right]$$

$$= \frac{\left(1-(-1)^n\right)}{\pi n}\left[\frac{4}{4-n^2}\right] \quad . \quad . \quad . \quad . \quad (22)$$

Now we solve (21):

$$4\pi b_n = \frac{2}{\pi}\int_0^{\pi}(1-\cos(x))\sin(nx)\,dx \quad \text{\textbackslash\textbackslash}$$

$$\text{\textbackslash\textbackslash} \Leftrightarrow b_n = \frac{1}{2\pi n}\int_0^{\pi}\left[\sin(nx) - \cos(x)\sin(nx)\right]dx$$

$$= \frac{1}{2\pi n}\left[\int_0^{\pi}\sin(nx)\,dx - \int_0^{\pi}\cos(x)\sin(nx)\,dx\right]$$

$$= \frac{1}{2\pi n}\left[-\frac{\cos(nx)}{n}\Bigg|_0^{\pi} - \frac{n}{m^2-n^2}\left((-1)^{n+m}-1\right)\Bigg|_{m=1}\right]$$

$$= \frac{1}{2\pi n}\left[\frac{1-(-1)^n}{n} + \frac{n\left(1-(-1)^{n+1}\right)}{1-n^2}\right]$$

$$\therefore\ b_n = \frac{1}{2\pi n}\left[\frac{1 - \cancel{n^2} - (-1)^n + (-1)^n n^2 + \cancel{n^2} + (-1)^n n^2}{n(1-n^2)}\right]$$

$$= \frac{1 + (-1)^n (n^2 + n^2 - 1)}{2\pi n^2 (1-n^2)} = \frac{1 + (-1)^n (2n^2 - 1)}{2\pi n^2 (1-n^2)} \quad \ldots \ (23)$$

Now consider $n = 1$:

$\to 0$ (see p $\boxed{8a}$)

$$b_1 = \frac{1}{2\pi}\left[\int_0^{\pi} \sin(x)\,dx - \int_0^{\pi} \cos(x)\sin(x)\,dx\right]$$

$$= -\frac{\cos(x)}{2\pi}\Bigg|_0^{\pi} = -\frac{1}{2\pi}\big((-1) - 1\big)$$

$$= \frac{2}{2\pi} = \frac{1}{\pi} \quad \ldots\ldots \ (24)$$

So we have,

$$u_1(x,t) = \left[a_1 \cos(4t) + b_1 \sin(4t)\right]\sin(x)$$

$$= \left[\frac{8}{3\pi}\cos(4t) + \frac{1}{\pi}\sin(4t)\right]\sin(x)$$

$\boxed{10a}$

$$u_2(x,t) = \left[a_2 \cos(8t) + b_2 \sin(8t)\right] \sin(2x)$$

$$= \left[0 + \frac{1 + (8-1)}{2\pi(4)(1-4)} \sin(8t)\right] \sin(2x)$$

$$= -\frac{1}{3\pi} \sin(8t) \sin(2x)$$

$$\therefore \boxed{\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) + \sum_{n=3}^{\infty} u_n(x,t) \\ &= \left(\frac{8}{3\pi}\cos(4t) + \frac{1}{\pi}\sin(4t)\right)\sin(x) \\ &\quad - \frac{1}{3\pi}\sin(8t)\sin(2x) \\ &\quad + \sum_{n=3}^{\infty}\left(\frac{(1-(-1)^n)}{\pi n}\frac{4}{4-n^2}\cos(4nt) \right. \\ &\qquad \left. + \frac{(1+(-1)^n(2n^2-1))}{2\pi n^2(1-n^2)}\sin(4nt)\right)\sin(nx) \end{aligned}}$$

4. a) We know that the homogeneous wave equation on infinite domains admits solutions of the form

$$u(x,t) = A(x+3t) + B(x-3t). \quad \ldots\ldots\ldots (25)$$

Now we apply the ICs: First:

$$u(x,0) = \begin{cases} \cos(\pi x) & \text{if } -1 < x < 1 \\ 0 & \text{else} \end{cases} \quad \Leftrightarrow \text{by } \tfrac{1}{\!}$$

$$\Leftrightarrow \quad A(x) + B(x) = \begin{cases} \cos(\pi x) & \text{if } -1 < x < 1 \\ 0 & \text{else} \end{cases} \quad \ldots (26)$$

Second:

$$\frac{\partial u}{\partial t}(x,0) = 0 \Leftrightarrow 3A'(x) - 3B'(x) = 0$$

$$\Leftrightarrow A'(x) - B'(x) = 0 \qquad (27)$$

<u>Case 1</u>: <u>$-1 < x < 1$</u>

(26) & (27) give us

$$\begin{cases} A(x) + B(x) = \cos(\pi x) \\ A'(x) - B'(x) = 0 \end{cases} \Leftrightarrow \begin{cases} A'(x) + B'(x) = -\pi \sin(\pi x) \\ A'(x) - B'(x) = 0 \end{cases}$$

cos/$\tfrac{1}{\!}$

$$\Leftrightarrow \begin{cases} 2A'(x) = -\pi \sin(\pi x) \\ 2B'(x) = -\pi \sin(\pi x) \end{cases} \Leftrightarrow \begin{cases} A'(x) = -\frac{\pi}{2} \sin(\pi x) \\ B'(x) = -\frac{\pi}{2} \sin(\pi x) \end{cases}$$

$$\Leftrightarrow \begin{cases} A(x) = \frac{1}{2} \cos(\pi x) \\ B(x) = \frac{1}{2} \cos(\pi x) \end{cases}$$

<u>Case 2:</u> $\underline{|x| > 1}$

(26) & (27) give us

$$\begin{cases} A(x) + B(x) = 0 \\ A'(x) - B'(x) = 0 \end{cases} \Rightarrow \begin{cases} A'(x) + B'(x) = 0 \\ A'(x) - B'(x) = 0 \end{cases} \Leftrightarrow \begin{cases} 2A'(x) = 0 \\ 2B'(x) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A(x) = C_1 \\ B(x) = C_2 \end{cases}$$

But $\because$ $A(x) + B(x) = 0$ we must have $C_2 = -C_1$

Putting cases 1 + 2 together, we have

$$u(x,t) = u_\ell(x,t) + u_r(x,t)$$

where

$$u_\ell(x,t) = \begin{cases} \frac{1}{2} \cos(\pi(x+3t)) & -1 < x+3t < 1 \\ 0 & \text{else} \end{cases}$$

and

$$u_r(x,t) = \begin{cases} \frac{1}{2}\cos(\pi(x-3t)) & -1 < x-3t < 1 \\ 0 & \text{else} \end{cases}$$

b) Plot

At $t=0$, $u(x,0) = \begin{cases} \cos(\pi x) & |x| < 1 \\ 0 & \text{else} \end{cases}$

At $t=1$, $u_l(x,1) = \frac{1}{2}\cos(\pi(x+3)) \quad -1 < x+3 < 1$

$\qquad\qquad 0 \quad \text{else}$

$$= \begin{cases} \frac{1}{2}\cos(\pi(x+3)) & -4 < x < -2 \\ 0 & \text{else} \end{cases}$$

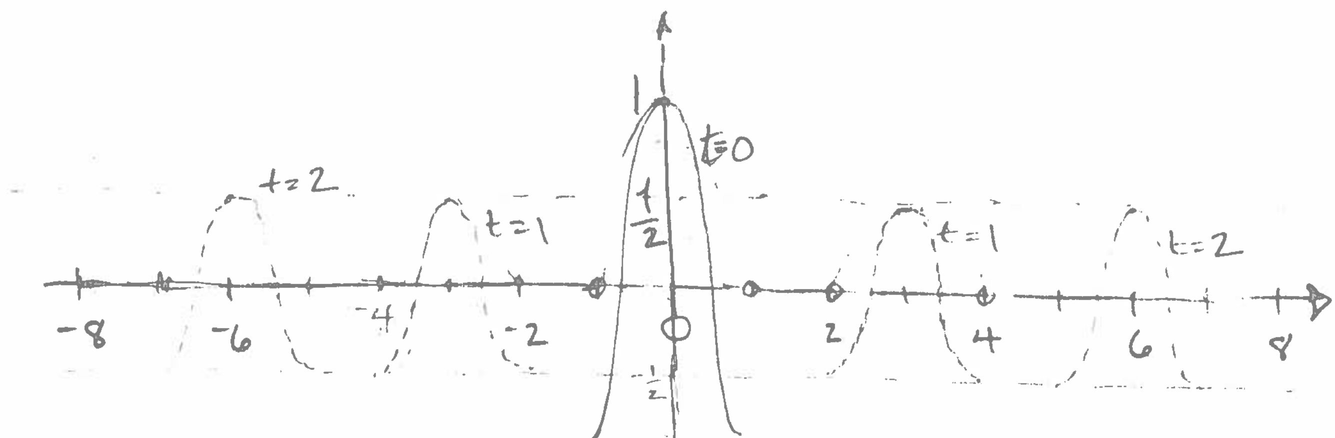
$$u_r(x,1) = \begin{cases} \frac{1}{2}\cos(\pi(x-3)) & -1 < x-3 < 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\cos(\pi(x-3)) & 2 < x < 4 \\ 0 & \text{else} \end{cases}$$

$$u(x,1) = u_l(x,1) + u_r(x,1)$$

at $t=2$, $$u_\ell(x,2) = \begin{cases} \frac{1}{2}\cos(\pi(x+6)) & -7<x<-5 \\ 0 & \text{else} \end{cases}$$

$$u_r(x,t) = \begin{cases} \frac{1}{2}\cos(\pi(x-6)) & 5<x<7 \\ 0 & \text{else} \end{cases}$$



The solution is two half-cosine waves travelling left & right at speed 3. At $t=0$ the two half cosine waves are summed & so the amplitude is double.

5. $u(x,t) = \sum_{n=0}^{\infty} c_n \cos(nx)\, e^{-\alpha n^2 t}$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \cos(nx) = 2\cos(5x) - \frac{7}{3}\cos(8x) + x^2$$

We can split the function $f(x)$ into two parts:

$$\begin{cases} f_1(x) = 2\cos(5x) - \frac{7}{3}\cos(8x) \\ f_2(x) = x^2 \end{cases}$$

Then the coefficients for the cosine series of $f_1(x)$ are obvious, & we only need to find the coefficients for the cosine series of $f_2(x)$.

$$f_2(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \frac{2\pi^2}{3}$$

$$\underset{n\in\mathbb{N}}{a_n} = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$\boxed{16}$

$$\frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = I \qquad u = x^2 \quad du = 2x\,dx \qquad dv = \cos(nx)\,dx \quad v = \frac{1}{n}\sin(nx)$$

$$I = \frac{2}{\pi}\left[\cancel{\frac{x^2}{n}\sin(nx)\Big|_0^{\pi}}^{\,0} - \int_0^{\pi}\frac{2x}{n}\sin(nx)\,dx\right]$$

$$= -\frac{4}{n\pi}\int_0^{\pi} x\sin(nx)\,dx \qquad u = x \quad du = dx \qquad dv = \sin(nx)\,dx \quad v = -\frac{1}{n}\cos(nx)$$

$$= -\frac{4}{n\pi}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right]$$

$$= -\frac{4}{n\pi}\left[\frac{\pi(-1)^{n+1}}{n} + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right] = \frac{4(-1)^n}{n^2}$$

$$\therefore\ \boxed{u(x,t) = 2\cos(5x) - \frac{7}{3}\cos(8x) + \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)e^{-\alpha n^2 t}}$$

6. The heat equation in polar coordinates is

$$\frac{\partial u}{\partial t} = D\nabla^2 u ,$$

where

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} .$$

The steady-state heat distribution is given by

$$\nabla^2 u = 0 \iff \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad \ldots \quad (28)$$

with BCs given by Figure 1. We assume $u(\theta, r) = T(\theta)R(r)$. Then (28) becomes

$$\frac{r^2 R''}{R} + r\frac{R'}{R} = -\frac{T''}{T} = \lambda \quad \ldots \quad (29)$$

Equation (29) gives rise to two ODE problems:

$$(B)\begin{cases} r^2 R'' + rR' - \lambda R = 0 \\ R(0) \text{ finite} \end{cases} \qquad (A)\begin{cases} T'' + \lambda T = 0 \\ T(0) = T(\pi) = 0 \end{cases}$$

Problem (A) is a BVP with solutions

$$T_n(\theta) = \sin(n\theta), \quad \lambda = n^2 \quad \ldots \quad (30)$$
$$n \in \mathbb{N}$$

Thus, problem (B) becomes

$$r^2 R'' + r R' - n^2 R = 0. \quad \cdots\cdots \quad (31)$$

We try solutions of the form r^{μ}. Then (31) becomes

$$r^2 \mu(\mu-1) r^{\mu-2} + r\mu r^{\mu-1} - n^2 r^{\mu} = 0 \Leftrightarrow$$

$$\Leftrightarrow \mu(\mu-1) + \mu - n^2 = 0$$

$$\Leftrightarrow \mu^2 - n^2 = 0 \Leftrightarrow \mu = \pm n$$

$$\therefore R_n(r) = a_n r^n + b_n r^{-n} \quad \cdots\cdots \quad (32)$$

Applying the condition that $R_n(0)$ be finite, we deduce that $b_n = 0$ & so (32) becomes

$$R_n(r) = a_n r^n \quad \cdots\cdots \quad (33)$$

With (30) & (33) we thus arrive at

$$u(\theta, r) = \sum_{n=1}^{\infty} \sin(n\theta)\, a_n r^n \quad \cdots\cdots \quad (34)$$

Now we apply the BC at $r = c$, and rewrite (34) as follows:

$$u(\theta, r) = \sum_{n=1}^{\infty} \sin(n\theta)\, b_n \left(\frac{r}{c}\right)^n$$

$$\therefore\ u(\theta, c) = u_0 \Leftrightarrow u_0 = \sum_{n=1}^{\infty} \sin(n\theta)\, b_n \left(\frac{c}{c}\right)^n$$

$$\Leftrightarrow u_0 = \sum_{n=1}^{\infty} b_n \sin(n\theta) \quad \ldots (35)$$

(35) is a Fourier sine series & so

$$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin(n\theta)\, d\theta = \frac{2}{\pi} u_0 \left(-\frac{1}{n}\right)\cos(n\theta)\Big|_0^{\pi}$$

$$= \frac{-2u_0}{n\pi}\left[\cos(n\pi) - 1\right] = \frac{2u_0}{n\pi}\left[1 - (-1)^n\right]$$

$$\therefore\ \boxed{u(\theta, r) = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}\left(1 - (-1)^n\right)\left(\frac{r}{c}\right)^n \sin(n\theta)}$$