Math 319
Assignment #5 - Solutions

1. We write

$$f(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x)$$

where

$$a_n = \int_1^e f(x)\,\Phi_n(x)\,r(x)\,dx$$

$$= \int_1^e x\,\sqrt{2}\cos\left(\left(\frac{2n-1}{2}\right)\pi\ln(x)\right)\frac{1}{x}\,dx$$

let $w = \ln(x)$, $dw = \frac{1}{x}\,dx$

$$= \int_0^1 e^{w}\sqrt{2}\cos\left(\left(\frac{2n-1}{2}\right)\pi w\right)dw = \sqrt{2}\,I$$

let $u = \cos\left(\frac{(2n-1)}{2}\pi w\right)$ $\quad du = -\left(\frac{2n-1}{2}\right)\pi\sin\left(\frac{(2n-1)}{2}\pi w\right)$,

$dv = e^{w}\,dw \quad v = e^{w}$

Then

$$I = \int_0^1 e^{w} \cos\left(\frac{(2n-1)}{2}\pi w\right) dw$$

$$= e^{w} \cos\left(\frac{(2n-1)}{2}\pi w\right)\Big|_0^1 + \int_0^1 e^{w} \frac{(2n-1)}{2}\pi \sin\left(\frac{(2n-1)}{2}\pi w\right) dw$$

let $u = \sin\left(\frac{(2n-1)}{2}\pi w\right)$ $\quad du = \frac{(2n-1)}{2}\pi \cos\left(\frac{(2n-1)}{2}\pi w\right)$

$dv = e^{w}\,dw \qquad v = e^{w}$

$$= \cancel{e \cos\left(\frac{(2n-1)}{2}\pi\right)}^{\,0} - 1\cos(0) + \frac{(2n-1)}{2}\pi \left[e^{w} \sin\left(\frac{(2n-1)}{2}\pi w\right)\Big|_0^1 - \int_0^1 e^{w} \frac{(2n-1)}{2}\pi \cos\left(\frac{(2n-1)}{2}\pi w\right) dw \right]$$

$$= -1 + \frac{(2n-1)}{2}\pi \left[e \sin\left(\frac{(2n-1)\pi}{2}\right) - \cancel{1 \cdot \sin(0)}^{\,0} - \frac{(2n-1)}{2}\pi\, I \right]$$

Solving for I we find

$$I + \frac{(2n-1)^2}{4}\pi^2 I = -1 + \frac{(2n-1)}{2} e^{\pi} \sin\left(\frac{(2n-1)\pi}{2}\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{4+(2n-1)^2\pi^2}{4}\right) I = -1 + \frac{(2n-1)}{2} e^{\pi} (-1)^{n-1}$$

$$\Leftrightarrow I = \frac{2\left(-2+(2n-1)e^{\pi}(-1)^{n-1}\right)}{4+(2n-1)^2\pi^2}$$

$$\therefore a_n = \frac{2\sqrt{2}\left(-2+(2n-1)e^{\pi}(-1)^{n-1}\right)}{4+(2n-1)^2\pi^2}$$

$$\therefore f(x) = x = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

$$= \sum_{n=1}^{\infty} \frac{2\sqrt{2}\left(-2+(2n-1)e^{\pi}(-1)^{n-1}\right)}{4+(2n-1)^2\pi^2} \sqrt{2} \cos\left(\frac{(2n-1)\pi \ln(x)}{2}\right)$$

$$\underline{= \sum_{n=1}^{\infty} \frac{4\left(-2+(2n-1)e^{\pi}(-1)^{n-1}\right)}{4+(2n-1)^2\pi^2} \cos\left(\frac{(2n-1)\pi \ln(x)}{2}\right)}$$

2. §11.3 #5

$$x^2y'' + xy' + \lambda y = 0 \quad . \; . \; . \; . \; . \quad (1)$$

A Sturm-Liouville equation has the form

$$(p(x)y'(x))' + q(x)y(x) + \lambda r(x)y(x) = 0 \quad . \; . \quad (2)$$

In order to match (2) & (1) we seek an integrating factor $\mu(x)$ such that

$$\mu(x)x^2y'' + \mu(x)xy' = (\mu(x)x^2y')' \iff 1$$

$$1 \iff \cancel{\mu(x)x^2}y'' + \mu(x)xy' = \cancel{\mu(x)x^2y''} + (\mu(x)x^2)'y'$$

$$\iff \mu(x)x = 2x\mu(x) + x^2\mu'(x)$$

$$\iff x^2\mu'(x) = -x\mu(x) \iff \frac{d\mu}{dx} = -\frac{1}{x}\mu$$

$$\iff \int \frac{d\mu}{\mu} = \int -\frac{1}{x}\,dx \iff \ln|\mu| = -\ln|x| + \cancel{c}^{\,0}$$

$\because x > 0$ & we only need a particular μ, we take

$$\ln(\mu) = -\ln(x) \iff \ln(\mu) = \ln\left(\frac{1}{x}\right) \iff \mu = \frac{1}{x}$$

Applying $\mu(x) = \frac{1}{x}$ to (1) we obtain

$$xy'' + y' + \frac{\lambda}{x} y = 0 \iff$$

$$\iff (xy')' + \frac{\lambda}{x} y = 0$$

$$\therefore \boxed{p(x) = x,\ q(x) = 0 \text{ and } r(x) = \frac{1}{x}}$$

Aside

we also observe that

- $p(x) > 0$ for $x > 0$ & $r(x) > 0$ for $x > 0$
- $p(x), p'(x), q(x)$ & $r(x)$ all C^0 on $[0, \infty)$

which would make this a Regular Sturm-Liouville problem.

1.

3. §11.3 #11

$$L[y] = (1+x^2)y'' + 2xy' + \lambda r y \quad \ldots\ldots \quad (3)$$

with

$$y'(0) = 0, \quad y(\pi) - (1+\pi^2)y'(\pi) = 0 \quad \ldots\ldots \quad (4)$$

To be self-adjoint, the operator L must satisfy

$$\int_0^{\pi} \left(uL[v] - vL[u]\right)(x)\,dx = 0 \quad \Leftrightarrow \quad //$$

$$// \Leftrightarrow \int_0^{\pi} \left(u\left[(1+x^2)v'' + 2xv' + \lambda r v\right] - v\left[(1+x^2)u'' + 2xu' + \lambda r u\right]\right)dx =$$

$$\Leftrightarrow \int_0^{\pi} \left((1+x^2)uv'' + 2xuv' - (1+x^2)vu'' - 2xvu'\right)dx = 0$$

$$\Leftrightarrow \int_0^{\pi} \left(u\left((1+x^2)v'\right)' - v\left((1+x^2)u'\right)'\right)dx = 0$$

$$\Leftrightarrow \int_0^{\pi} \Big(\left[u(1+x^2)v'\right]' - u'(1+x^2)v' - \left[v(1+x^2)u'\right]'$$

$$+ v'(1+x^2)u' \Big)\,dx = 0 \quad \Leftrightarrow \quad //$$

(I) $\Leftrightarrow$ $\displaystyle\int_0^{\pi}\left(\left[u(1+x^2)v'\right]' - \left[v(1+x^2)u'\right]'\right)dx = 0$

$\Leftrightarrow$ $\left[u(1+x^2)v' - v(1+x^2)u'\right]\Big|_0^{\pi} = 0$

$\Leftrightarrow$ $u(\pi)(1+\pi^2)v'(\pi) - u(0)v'(0)$

$- v(\pi)(1+\pi^2)u'(\pi) + v(0)u'(0) = 0$ (II)

We now apply the BCs (2) + obtain

(II) $\Leftrightarrow$ $u(\pi)(1+\pi^2)v'(\pi) - 0 - v(\pi)(1+\pi^2)u'(\pi)$

$+ 0 = 0$

$\Leftrightarrow$ $u(\pi)v(\pi) - v(\pi)u(\pi) = 0$

Which is a true statement. $\therefore$ the operator L defined in (1) + (2) is self-adjoint.

end of proof.

A. §11.3 #12(c)

$$LHS = (f+h, g)$$

$$= \int_a^b (f(x) + h(x))\, g(x)\, dx$$

$$= \int_a^b [f(x)g(x) + h(x)g(x)]\, dx$$

$$= \int_a^b f(x)g(x)\, dx + \int_a^b h(x)g(x)\, dx$$

$$= (f,g) + (h,g) = RHS$$

end of proof.

5. §11.3 #20

$$y'' - 2y + \lambda y = 0 \quad \dots\dots\dots\dots (3)$$
$$y(0) = 0, \; y'(\pi) = 0 \quad \dots\dots\dots\dots (4)$$

This is a Regular Sturm - Liouville BVP with

$$\left.\begin{array}{l} p(x) = 1 \;, \; p'(x) = 0 \\ q(x) = -2 \;, \; r(x) = 1 \end{array}\right\} \text{all continuous on } [0, \pi]$$

and $p(x) > 0$ & $r(x) > 0$ on $[0, \pi]$. Also

$$a_1 = 1 \;, a_2 = 0 \;, \; b_1 = 0 \;, b_2 = 1,$$

So at least one of $a_1 + a_2$, and $b_1 + b_2$ is nonzero.

The solutions to Sturm - Liouville problems are orthogonal eigenfunctions. To solve (3), consider the characteristic equation:

$$\rho^2 - 2 + \lambda = 0 \Leftrightarrow \rho^2 = 2 - \lambda \Leftrightarrow \rho = \pm\sqrt{2 - \lambda} \quad \dots (5)$$

Case 1 $\quad 2-\lambda > 0 \Leftrightarrow 2-\lambda = \mu^2$

then $y(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$

apply BCs:

$$\begin{cases} y(0) = 0 \Leftrightarrow c_1 = 0 \\ y'(\pi) = 0 \Leftrightarrow \mu c_2 \cosh(\mu\pi) = 0 \Leftrightarrow c_2 = 0 \end{cases}$$

$\therefore$ we obtain only the trivial solution

Case 2 $\quad 2-\lambda = 0 \Leftrightarrow \lambda = 2$

then $y(x) = c_1 x + c_2$

apply BCs.

$$\begin{cases} y(0) = 0 \Leftrightarrow c_2 = 0 \\ y'(\pi) = 0 \Leftrightarrow c_1 = 0 \end{cases}$$

$\therefore$ we obtain only the trivial solution

Case 3 $\quad 2-\lambda < 0 \Leftrightarrow 2-\lambda = -\mu^2$

$\therefore$ We set

$$2-\lambda = -\mu^2 \Leftrightarrow \lambda = 2+\mu^2 \quad , \quad \mu > 0 \ldots (6)$$

In this case, the solutions of (3) are

$$y(x) = c_1 \cos\left(\left(2-(2+\mu^2)\right)x\right) + c_2 \sin\left(\left(2-(2+\mu^2)\right)x\right)$$

$$= c_1 \cos(\mu x) + c_2 \sin(\mu x) \quad \ldots\ldots \quad (7)$$

Plugging the BCs (4) into (7) we obtain

$$y(0) = 0 \Leftrightarrow c_1 \cos(0) + c_2 \sin(0) = 0$$

$$\Leftrightarrow c_1 = 0 \quad \ldots\ldots\ldots \quad (8)$$

$$y'(\pi) = 0 \Leftrightarrow \mu c_2 \cos(\mu\pi) = 0 \quad \ldots\ldots\ldots \quad (9)$$

For nontrivial solutions we require

$$\cos(\mu\pi) = 0 \Leftrightarrow \mu\pi = (2n-1)\frac{\pi}{2}, \quad n \in \mathbb{N}$$

$$\Leftrightarrow \mu = \frac{(2n-1)}{2}, \quad n \in \mathbb{N} \quad \ldots \quad (10)$$

With (8), (9) & (10) we have

$$\text{eigenvalues: } \lambda = 2 + \frac{(2n-1)^2}{4}, \quad n \in \mathbb{N}$$

$$\text{eigenfunctions: } y_n(x) = \sin\left(\frac{(2n-1)}{2}x\right)$$

We now normalize these eigenfunctions to create an orthonormal set. The orthonormal functions $\Phi_n(x)$ are found from

$$\Phi_n(x) = c_n y_n(x), \quad c_n > 0$$

where

$$\int_0^{\pi} c_n^2 y_n^2(x)\,dx = 1 \iff //$$

$$// \iff \int_0^{\pi} c_n^2 \sin^2\left(\frac{(2n-1)}{2}x\right) dx$$

(Note: the weight fn $r(x) = 1$, see p$\boxed{9}$)

$$\Leftrightarrow c_n^2 \int_0^{\pi} \frac{\left(1 - \cos((2n-1)x)\right)}{2}\, dx = 1$$

$$\Leftrightarrow c_n^2 \left[\frac{x}{2} - \frac{1}{2(2n-1)} \sin((2n-1)x) \right]_0^{\pi} = 1$$

$$\Leftrightarrow c_n^2 \left[\frac{\pi}{2} - \frac{1}{2(2n-1)} \cancelto{0}{\sin((2n-1)\pi)} \right] = 1$$

$$\Leftrightarrow c_n^2 \frac{\pi}{2} = 1 \Leftrightarrow c_n^2 = \frac{2}{\pi} \Leftrightarrow c_n = \sqrt{\frac{2}{\pi}}$$

$\therefore$ The normalized eigenfunctions are

$$\Phi_n(x) = \sqrt{\frac{2}{\pi}} \sin\left(\frac{(2n-1)}{2} x\right) \quad \ldots \quad (11)$$

We now seek to express $f(x) = x$ in an eigenfunction expansion using (11).

We write

$$x = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{\pi}} \sin\left(\left(\frac{2n-1}{2}\right) x\right) \quad \ldots \quad (12)$$

where

$$a_n = \int_0^{\pi} x \sqrt{\frac{2}{\pi}} \sin\left(\left(\frac{2n-1}{2}\right) x\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} x \sin\left(\left(\frac{2n-1}{2}\right) x\right) dx$$

let $u = x$ $\qquad du = dx$

$dv = \sin\left(\left(\frac{2n-1}{2}\right) x\right) dx \qquad v = \frac{-2}{2n-1} \cos\left(\left(\frac{2n-1}{2}\right) x\right)$

$$= \sqrt{\frac{2}{\pi}} \left[\cancelto{0}{\frac{-2x}{2n-1} \cos\left(\left(\frac{2n-1}{2}\right) x\right) \Bigg|_0^{\pi}} + \int_0^{\pi} \frac{2}{2n-1} \cos\left(\left(\frac{2n-1}{2}\right) x\right) dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{+4}{(2n-1)^2} \sin\left(\left(\frac{2n-1}{2}\right) x\right) \Bigg|_0^{\pi} \right]$$

Note that

n	$(2n-1)$	$\sin\left(\frac{(2n-1)\pi}{2}\right)$	$(-1)^{n-1}$
1	1	$\sin\left(\frac{\pi}{2}\right) = 1$	$(-1)^0 = 1$
2	3	$\sin\left(\frac{3\pi}{2}\right) = -1$	$(-1)^1 = -1$
3	5	$\sin\left(\frac{5\pi}{2}\right) = 1$	$(-1)^2 = 1$
4	7	$\sin\left(\frac{7\pi}{2}\right) = -1$	$(-1)^3 = -1$

$\therefore$ we can write

$$a_n = \sqrt{\frac{2}{\pi}} \left(\frac{+4}{(2n-1)^2}\right) \left[(-1)^{n-1} - 0\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{(+4)}{(2n-1)^2} (-1)^{n-1} \quad \ldots\ldots \quad (13)$$

Combining (12) + (13) we have

$$x = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{4}{(2n-1)^2} (-1)^{n-1} \sqrt{\frac{2}{\pi}} \sin\left(\frac{(2n-1)}{2} x\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{4}{(2n-1)^2} (-1)^{n-1} \sin\left(\frac{(2n-1)}{2} x\right)$$

$$\boxed{x = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\left(\frac{(2n-1)}{2} x\right)}$$

6. $\begin{cases} x^2y'' + 3xy' + 2y = h \\ y'(1) = 0,\ y'(e^{\pi}) = 0 \end{cases}$

a) We first find the adjoint BVP.

$$A_2 = x^2,\quad A_1 = 3x,\quad A_0 = 2$$

So

$$L^+[y] := (A_2 y)'' - (A_1 y)' + A_0 y$$

$$= (x^2 y)'' - (3xy)' + 2y$$

$$= (x^2y' + 2xy)' - 3xy' - 3y + 2y$$

$$= x^2y'' + 2xy' + 2xy' + 2y - 3xy' - y$$

$$= x^2y'' + xy' + y$$

Now we find $B^+[y]$ such that

$$P(u,v)\Big|_{x=1}^{x=e^{\pi}} := \left(uA_1v - u(A_2v)' + u'A_2v\right)\Big|_{x=1}^{x=e^{\pi}} = 0$$

where $B[u] = 0$ + $L^+[v] = 0$. Plugging in A_2 + A_1 we obtain:

$$P(u,v)\Big|_1^{e^\pi}=0 \Leftrightarrow \Big[3xuv-u(x^2v)'+u'x^2v\Big]_{x=1}^{x=e^\pi}=0$$

$$\Leftrightarrow \Big[3xuv-u[x^2v'+2xv]+u'x^2v\Big]_{x=1}^{x=e^\pi}=0$$

$$\Leftrightarrow \Big(xuv+x^2[u'v-uv']\Big)\Big|_{x=1}^{x=e^\pi}=0$$

$$\Leftrightarrow e^\pi u(e^\pi)v(e^\pi)+e^{2\pi}\Big[\cancelto{0}{u'(e^\pi)}v(e^\pi)-u(e^\pi)v'(e^\pi)\Big]$$
$$-u(1)v(1)-\Big[\cancelto{0}{u'(1)}v(1)-u(1)v'(1)\Big]=0$$

$$\Leftrightarrow e^\pi u(e^\pi)\Big[v(e^\pi)-e^{\pi}v'(e^\pi)\Big]$$
$$-u(1)\Big[v(1)-v'(1)\Big]=0$$

$\because$ $u(e^\pi)$ + $u(1)$ are arbitrary, we must have

$$\begin{cases} v(1)-v'(1)=0 \\ v(e^\pi)-e^\pi v'(e^\pi)=0 \end{cases}$$

$\therefore$ the adjoint BVP is

$$x^2y''+xy'+y=0 \quad \text{with} \begin{cases} v(1)-v'(1)=0 \\ v(e^\pi)-e^\pi v'(e^\pi)=0 \end{cases}$$

b) Determine under what conditions sol'ns to the nonhomogeneous BVP exist.

We will use the Fredholm Alternative to answer this question. We need the solutions of the adjoint BVP.

$$x^2y'' + xy' + y = 0$$

char. eqn: (plug in $y = x^r$)

$$r(r-1) + r + 1 = 0 \Leftrightarrow r^2 - r + r + 1 = 0 \Leftrightarrow r^2 + 1 = 0$$
$$\Leftrightarrow r^2 = -1 \Leftrightarrow r = \pm i$$

$\therefore$ sol'ns are

$$y(x) = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$
$$y'(x) = -\frac{c_1}{x}\sin(\ln(x)) + \frac{c_2}{x}\cos(\ln(x))$$

Apply the BCs:

$$y(1) - y'(1) = 0 \Leftrightarrow c_1\cos(\ln(1)) + \overbrace{c_2\sin(\ln(1))}^{0}$$
$$+ \overbrace{\frac{c_1}{1}\sin(\ln(1))}^{0} - \frac{c_2}{1}\cos(\ln(1)) =$$

$$\Leftrightarrow c_1 - c_2 = 0 \Leftrightarrow c_1 = c_2$$

$$y(e^{\pi}) - e^{\pi} y'(e^{\pi}) = 0 \Leftrightarrow c_1 \cos(\ln(e^{\pi})) + c_2 \sin(\ln(e^{\pi})) + \cancel{e^{\pi}} \frac{c_1}{\cancel{e^{\pi}}} \sin(\ln(e^{\pi})) - \cancel{e^{\pi}} \frac{c_2}{\cancel{e^{\pi}}} \cos(\ln(e^{\pi})) = 0$$

$$\Leftrightarrow c_1 \cos(\pi) + c_1 \cancel{\sin(\pi)}^{0} + c_1 \cancel{\sin(\pi)}^{0} - c_1 \cos(\pi) = 0$$

$$\Leftrightarrow 0 = 0 \quad \text{true}$$

$\therefore$ c_1 is arbitrary & we have

$$y(x) = c_1 \left[\cos(\ln(x)) + \sin(\ln(x))\right]$$

For solutions to exist, we require

$$\int_1^{e^{\pi}} h(x) \left[\cos(\ln(x)) + \sin(\ln(x))\right] dx = 0$$