UBC Okanagan
The Irving K. Barber School of Arts and Sciences

FINAL EXAMINATION
Math 319 - Intro to Partial Differential Equations

Monday, December 12, 2011 Time: 3 hours

You are being evaluated on the presentation, as well as the correctness, of your answers. Show your work clearly, include explanations and, where appropriate, references to theorems.

Useful Identities:

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ $\qquad 2\sin x \cos y = \sin(x+y) + \sin(x-y)$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$ $\qquad 2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$ $\qquad 2\cos x \cos y = \cos(x-y) + \cos(x+y)$

$\sin 2x = 2\sin x \cos x$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

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1. Compute the Fourier series for

$$f(x) = \begin{cases} 0 & -\pi \le x < 0, \\ \sin x & 0 \le x \le \pi. \end{cases}$$

Discuss the convergence of the series, and show graphically the function represented by the series for all x.

2. Solve the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{on} \qquad 0 < x < 1, \quad 0 < y < 2,$$

$u_x(0, y) = u_x(1, y) = 0$ for $0 < y < 2$,

$u(x, 0) = f(x)$ and $u(x, 2) = 0$ for $0 < x < 1$.

3. Solve the following initial-boundary value problem for $u(x,t)$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{on} \qquad 0 < x < 1, \quad t > 0,$$

$u(0, t) = 1$ and $\frac{\partial u}{\partial x}(1, t) = 0$ for $t > 0$,

$u(x, 0) = 0$ for $0 < x < 1$.

4. Obtain all the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0 \ ;$$
$$y'(0) = 0,$$
$$y(1) - 2y'(1) = 0.$$

5. Convert the following equation into a Sturm-Liouville equation.

$$xy'' + (1 - x)y' + \lambda y = 0.$$

6. Consider the following boundary value problem

$$y'' + \lambda^2 y = 0,$$
$$y(0) = 0, \quad y'(\pi) = 0.$$

The eigenvalues are $\lambda_n = \frac{(2n-1)}{2}$ for $n = 1,\ 2,\ ...$ with corresponding eigenfunctions

$$\phi_n(x) = a_n \sin\left(\frac{2n-1}{2}\right) x.$$

Express $f(x) = x$ in terms of an eigenfunction expansion.

7. Consider the following boundary value problem:

$$x^2 y''(x) + 2xy'(x) = h(x);$$
$$y(1) = y(2), \quad y(1)' = y'(2).$$

(a) Find the adjoint boundary value problem for the associated homogeneous problem.

(b) State the Fredholm alternative.

(c) Determine condition(s) on $h(x)$ that guarantee that the given non-homogeneous boundary value problem has a solution.