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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 319
Date: Dec 12th, 2014 Time: 6:00pm Duration: 3 hours.
This exam has 7 questions for a total of 58 points.

NAME: ______________________________

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

This exam is an individual exam only. You have 3 hours to complete the exam.

Problem Number	1	2	3	4	5	6	7	**Total**
Points Earned								
Points Out Of	8	3	9	6	12	7	10	58

1. Consider the PDE

$$3\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} + u = x. \qquad (1)$$

2 (a) Find the equation for the characteristic lines of (1).

2 (b) Use a change of variables $v(w, z) = u(x, y)$ to convert the PDE problem (1) into the ODE problem

$$\frac{\partial v}{\partial z} - \frac{1}{2}v = \frac{1}{4}(3z - w). \qquad (2)$$

Show all of your work. (*Note: You may obtain* $v_z - (1/2)v = (1/4)(3z + w)$ *if you define your* w *differently.*)

4 (c) Solve (2). Give the final answer in terms of x and y.

[6] 2. Consider the BVP

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \lambda y = 0, \qquad 0 < x < 2, \tag{3a}$$

$$y(0) = 0, \quad \frac{dy}{dx}(2) = 0. \tag{3b}$$

Find the values of λ for which the given problem has nontrivial solutions. Check all cases and write down the eigenfunctions.

(Additional workspace for problem # 2.)

3. Consider

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0, \tag{4a}$$

$$u(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \qquad t > 0 \tag{4b}$$

$$u(x,0) = f(x), \qquad 0 < x < L. \tag{4c}$$

where

$$f(x) = \sin(4x) + 3\sin(6x) - \sin(10x). \tag{5}$$

4 (a) Apply separation of variables (use $F(x)$ and $G(t)$) to turn this PDE problem into a pair of ODE problems.

5 (b) Given that the eigenvalues and eigenfunctions of the BVP in part (a) are

$$\lambda_n = n^2, \quad n \in \mathbb{N}, \qquad F_n(x) = \sin(nx), \tag{6}$$

find the solution to the system (4). (*Note: This means that the hard part of "step 3" has been done for you!*)

(Additional workspace for problem # 3.)

4. Consider the function

$$f(x) = \begin{cases} -1, & 0 \le x < 1, \\ 2, & 1 \le x \le 3. \end{cases} \tag{7}$$

A plot of the function appears in Figure 1.

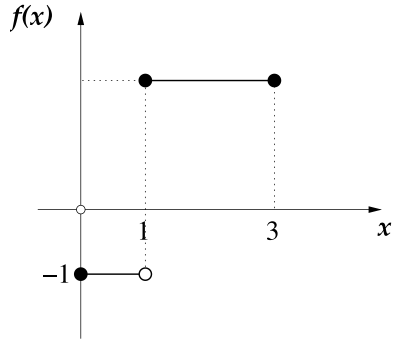

Figure 1: Plot of $f(x)$ for question 4.

2 (a) Let $F(x)$ be the function to which the fourier sine representation of $f(x)$ converges. Sketch $F(x)$ on the axes below. (*Note: This problem is harder than it looks! Take your time and do it carefully.*)

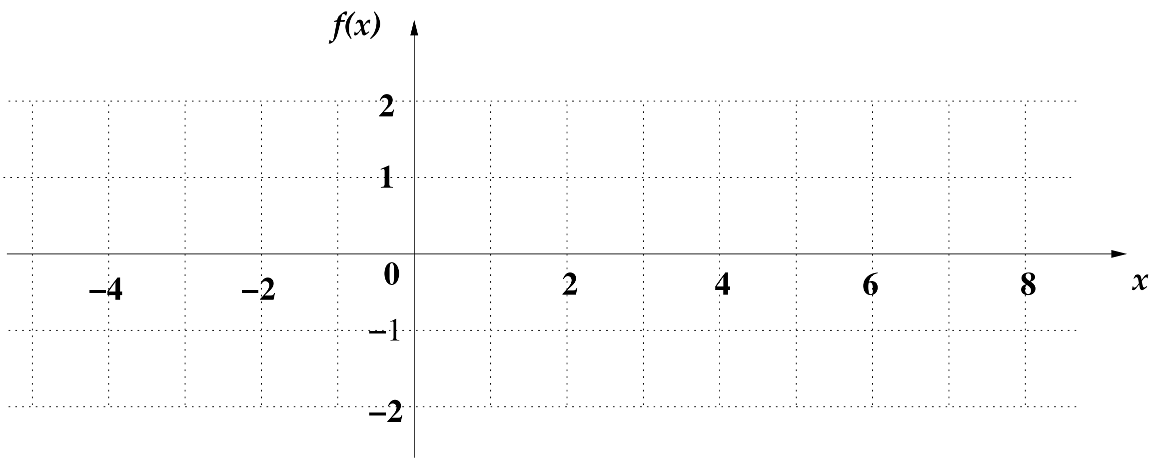

Figure 2: Axes for question 4(a).

4 (b) Let $G(x)$ be the function to which the fourier cosine representation of $f(x)$ converges. Compute $G(x)$.

12 5. Find a solution for the Neumann BVP on the disk

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < r < a, \quad -\pi \le \theta \le \pi, \tag{8a}$$

$$\frac{\partial u}{\partial r}(a, \theta) = f(\theta), \qquad -\pi \le \theta \le \pi. \tag{8b}$$

In solving the BVP in θ, you can assume that the eigenvalue is positive or zero. *Hint: Recall that $u(r, \theta)$ should be continuous at $\theta = -\pi$ and $\theta = \pi$, and that $u(r, \theta)$ must be finite at the origin.*

(Additional workspace for problem # 5.)

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6. Consider

$$(xy')' + \frac{\mu^2}{x} y = 0, \qquad 1 < x < e, \quad \mu > 0, \quad \mu \in \mathbb{R} \tag{9a}$$

$$y'(1) = 0, \quad y(e) = 0. \tag{9b}$$

2 (a) Verify that the problem (9) is a Regular Sturm-Liouville BVP and give the coefficients.

5 (b) Compute the solution.

10 7. Find the eigenfunction expansion for the solution to the following non-homogeneous BVP:

$$y'' + 2y = \pi x, \qquad 0 < x < \pi, \tag{10a}$$
$$y(0) = y(\pi) = 0. \tag{10b}$$

(Additional workspace for problem # 7.)

Information That May Be Useful

Method of Eigenfunction Expansions: (copied from p 679 of the text) To obtain an eigenfunction expansion for the solution ϕ to the nonhomogeneous regular Sturm-Liouville BVP

$$L[y] + \mu r y = f; \quad B[y] = 0,$$

follow the steps below:

- Find an orthogonal system of eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ for
$$L[y] + \lambda r y = 0, \quad B[y] = 0.$$
- Compute the eigenfunction expansion for f/r, that is, determine the coefficients γ_n so that
$$\frac{f}{r} = \sum_{n=1}^{\infty} \gamma_n \phi_n.$$
- If $\mu \neq \lambda_N$ for $n \in \mathbb{N}$, then the solution ϕ is given by
$$\phi = \sum_{n=1}^{\infty} \frac{\gamma_n}{\mu - \lambda_n} \phi_n.$$
- If $\mu = \lambda_N$ for some N and $\gamma_N \neq 0$, then there is no solution.
- If $\mu = \lambda_N$ for some N and $\gamma_N = 0$, then
$$\phi = c_N \phi_N + \sum_{n=1 n \neq N}^{\infty} \frac{\gamma_n}{\mu - \lambda_n} \phi_n$$
is a solutions for any choice of the parameter c_N.

Lagrange's Identity: (copied from p 669 of the text) Let L be the differential operator

$$L[y](x) := A_2(x) y''(x) + A_1(x) y'(x) + A_0(x) y(x),$$

and let L^+ be its formal adjoint

$$L^+[y] := (A_2 y)'' - (A_1 y)' + A_0 y.$$

Then,

$$L[u] v l - u L^+[v] = \frac{d}{dx} \left[P(u, v) \right],$$

where $P(u, v)$, the bilinear concomitant associated with L, is defined as

$$P(u, v) := u A_1 v - u (A_2 v)' + u' A_2 v.$$

The coefficient γ_n in the method of eigenfunction expansions:

$$\gamma_n = \frac{\int_a^b \frac{f(x)}{r(x)}\phi_n(x)r(x)dx}{\int_a^b \phi_n^2(x)r(x)dx}$$

Integrals that may be useful:

$$\int ze^{-az}dz = \frac{-1}{a}ze^{-az} + \frac{1}{a}\int e^{-az}dz$$

$$\int_0^\pi \sin^2(nx)dx = \frac{\pi}{2} - \frac{\sin(2\pi n)}{4n}$$

$$\int x\sin(nx)dx = \frac{\sin(nx) - nx\cos(nx)}{n^2}$$