a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 319
Date: Oct 2nd, 2014 Time: 12:30pm Duration: 45 minutes.
This exam has 4 questions for a total of 20 points.

NAME: Solutions

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

This is a two-stage exam. You have 45 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Problem Number	1	2	3	4	**Total**
Points Earned					
Points Out Of	5	5	6	4	20

$\boxed{5}$ 1. Use the method of characteristics to transform the PDE below into an ODE. Sketch the characteristic lines. *Note: Do not solve the ODE!!*

$$2u_x - 3u_y + x^2 u = \frac{1}{y} \tag{1}$$

Let

$$\begin{cases} w = -3x - 2y \\ z = y \end{cases} \Leftrightarrow \begin{cases} 3x = -2y - w \\ y = z \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{3}(2z + w) \\ y = z \end{cases}$$

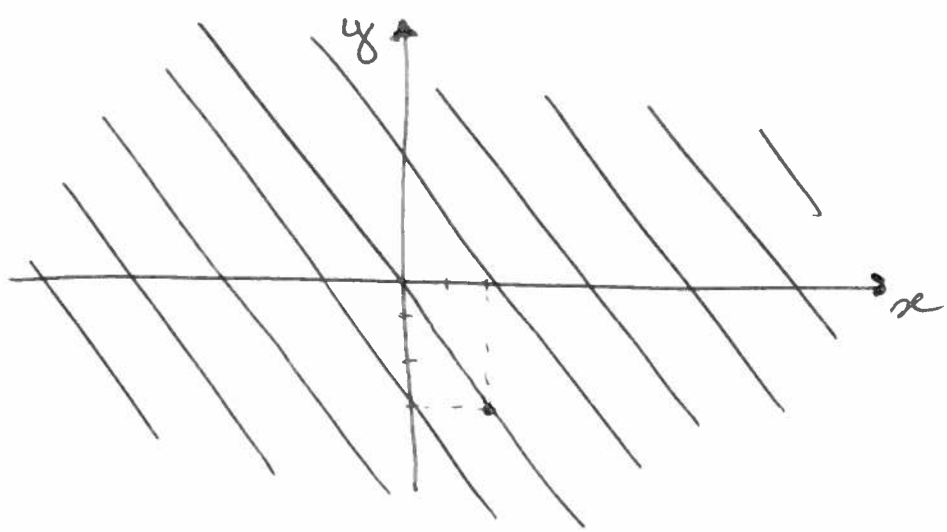
Then, if $v(w,z) = u(x,y)$,

$$2u_x - 3u_y + x^2 u = \frac{1}{y} \Leftrightarrow -3v_z + \left(-\frac{1}{3}\right)^2 (2z+w)^2 v = \frac{1}{z}$$

$$\Leftrightarrow v_z - \frac{1}{27}(2z+w)^2 v = \frac{-1}{3z}$$

Characteristic lines: lines of constant w.

$w = K \Leftrightarrow -3x - 2y = K \Leftrightarrow -2y = 3x + K \Leftrightarrow y = -\frac{3}{2}x + \tilde{K}$



$\boxed{5}$ 2. Use the technique of "separation of variables" to transform the PDE below into a pair of ODEs. At a critical point in the calculations, why can you set both sides equal to $-\lambda$? *(Note: Simply state the ODEs; do not solve them!)*

$$\frac{\partial^2 u}{\partial t^2} - k\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + \mu e^{ax} u \tag{2}$$

Note that k, μ, and a are constants.

Let $u(x,t) = F(x)G(t)$. Then (2) becomes

$$FG'' - kFG' = \alpha^2 F''G + \mu e^{ax} FG \iff /\!/$$

$$/\!/ \iff \frac{FG''}{\alpha^2 FG} - \frac{kFG'}{\alpha^2 FG} = \frac{\alpha^2 F''G}{\alpha^2 FG} + \frac{\mu e^{ax} FG}{\alpha^2 FG}$$

$$\iff \underbrace{\underbrace{\frac{G''}{\alpha^2 G} - \frac{kG'}{\alpha^2 G}}_{\text{depends on } t \text{ only}} = \underbrace{\frac{F''}{F} + \frac{\mu e^{ax}}{\alpha^2}}_{\text{depends on } x \text{ only}}}_{\therefore \text{ ea side must be equal to a cst.}}$$

We write:

$$\frac{G''}{\alpha^2 G} - \frac{kG'}{\alpha^2 G} = \frac{F''}{F} + \frac{\mu e^{ax}}{\alpha^2} = -\lambda$$

which gives us two ODEs:

$$G'' - kG' + \lambda\alpha^2 G = 0$$

$$F'' + \left(\frac{\mu e^{ax}}{\alpha^2} + \lambda\right) F = 0$$

$\boxed{6}$ 3. What are the eigenfunctions $F_n(x)$ and eigenvalues ρ of the BVP below? Assume $\rho \in \mathbb{R}$, $\rho > 0$.
Note: This assumption gives you just one case to consider.

$$F''(x) + 2F'(x) + (1+\rho^2)F(x) = 0, \qquad 0 < x < 2\pi, \tag{3a}$$
$$F(0) = F(2\pi) = 0. \tag{3b}$$

Characteristic equation:

$$r^2 + 2r + (1+\rho^2) = 0 \iff r = -1 \pm \sqrt{1-(1+\rho^2)}$$
$$\iff r = -1 \pm \sqrt{-\rho^2}$$
$$\iff r = -1 \pm i\rho$$

$$\therefore \quad F(x) = e^{-x}\left(c_1 \cos(\rho x) + c_2 \sin(\rho x)\right)$$

Now apply the BCs:

$$\begin{cases} F(0) = 0 \\ F(2\pi) = 0 \end{cases} \iff \begin{cases} c_1 = 0 \\ e^{-2\pi} c_2 \sin(2\pi\rho) = 0 \end{cases}$$

For nontrivial solutions we require

$$2\pi\rho = n\pi \iff \rho = \frac{n}{2}, \quad n = 1, 2, 3, \ldots$$

Thus, we have

eigenvalues: $\rho = \dfrac{n}{2}$

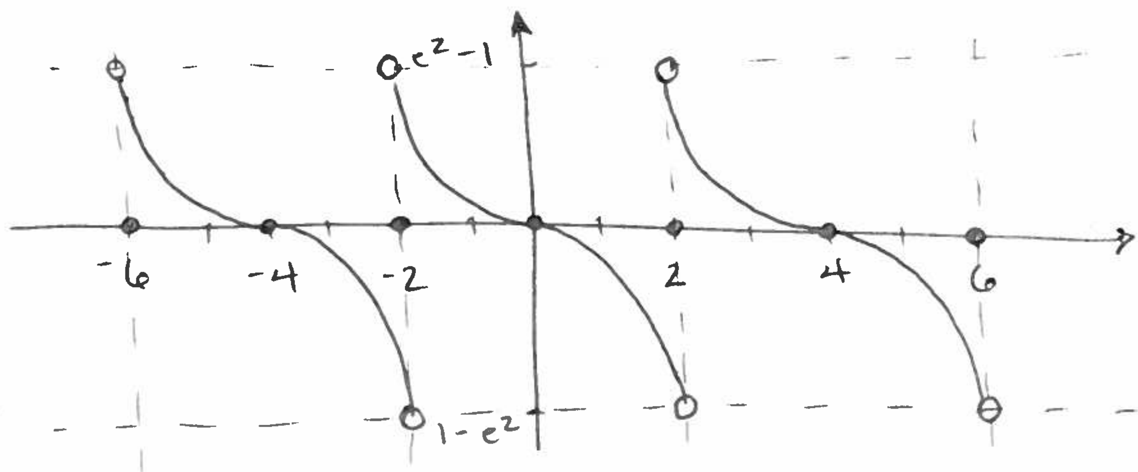
eigenfunctions: $F_n(x) = B_n e^{-x} \sin\left(\dfrac{nx}{2}\right)$

4. Consider the function

$$f(x) = 1 - e^{x}, \qquad 0 < x < 2. \tag{4}$$

A sketch of the function is shown in Figure 1 (last page of the test).

$\boxed{2}$ (a) On the interval $[-6, 6]$, sketch the function to which the Fourier sine approximation of $f(x)$ converges.



$\boxed{2}$ (b) The Fourier sine series of $f(x)$ is written

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\gamma x).$$

(a) What is γ?

$$\gamma = \frac{n\pi}{L} = \frac{n\pi}{2}$$

(b) Write the formula for the coefficients b_n. *Note: Do not solve for the coefficients!*

$$b_n = \frac{2}{2}\int_0^2 (1-e^{x}) \sin\left(\frac{n\pi x}{2}\right) dx$$

Solution to Bonus problem #4(b), Group Test

$$b_n = \int_0^2 (1 - e^x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx - \int_0^2 e^x \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \left[\frac{e^x}{\left(\frac{n\pi}{2}\right)^2+1}\left(\sin\left(\frac{n\pi x}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi x}{2}\right)\right)\right]_0^2$$

$$= -\frac{2}{n\pi}\left[\cos(n\pi) - \cancelto{1}{\cos(0)}\right] - \left[\frac{e^2}{\left(\frac{n\pi}{2}\right)^2+1}\left(\cancelto{0}{\sin(n\pi)} - \frac{n\pi}{2}\cos(n\pi)\right)\right.$$

$$\left. - \frac{1}{\left(\frac{n\pi}{2}\right)^2+1}\left(\cancelto{0}{\sin(0)} - \frac{n\pi}{2}\cancelto{1}{\cos(0)}\right)\right]$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - 1\right) + \frac{4e^2}{(n\pi)^2+4}\,\frac{n\pi}{2}\cos(n\pi) - \frac{4}{(n\pi)^2+4}\,\frac{n\pi}{2}$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - 1\right) + \frac{2n\pi}{(n\pi)^2+4}\left(e^2\cos(n\pi) - 1\right)$$

OR

$$= \frac{-2}{n\pi}\left((-1)^n - 1\right) + \frac{2n\pi}{(n\pi)^2+4}\left(e^2(-1)^n - 1\right)$$