D'Alembert Solutions - Two Worked Examples

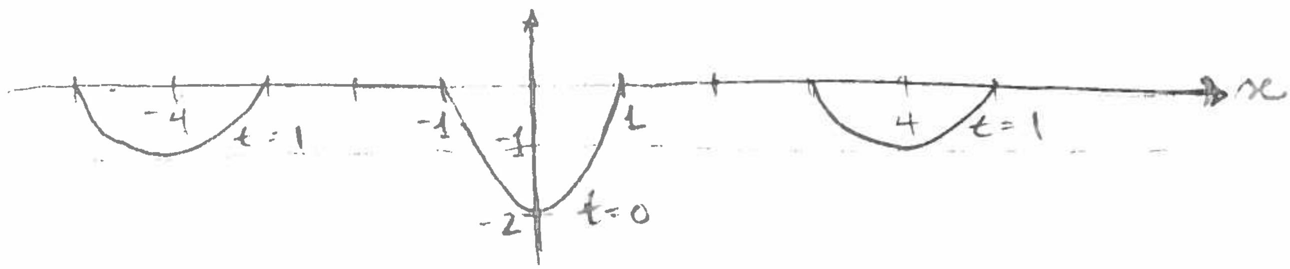
1. M2 Sample Problems, #9

$$\frac{\partial^2 u}{\partial t^2} = 4^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

$$u(x,0) = \begin{cases} 2(x^2-1) & \text{if } |x| < 1 \\ 0 & \text{else} \end{cases}$$

a)



b) In the special case where $u_t(x,0) = 0$, we know that

$$u(x,t) = \frac{1}{2}\left(f(x+4t) + f(x-4t)\right)$$

Where $u(x,0) = f(x)$. The solution $\frac{1}{2}f(x+4t)$ is a left-moving wave & $\frac{1}{2}f(x-4t)$ is a right-moving wave.

We have;

$$\frac{1}{2} f(x+4t) = \begin{cases} (x+4t)^2 - 1, & \text{if } |x+4t| < 1, \\ 0, & \text{else}, \end{cases}$$

$$\frac{1}{2} f(x-4t) = \begin{cases} (x-4t)^2 - 1, & \text{if } |x-4t| < 1, \\ 0, & \text{else}, \end{cases}$$

and

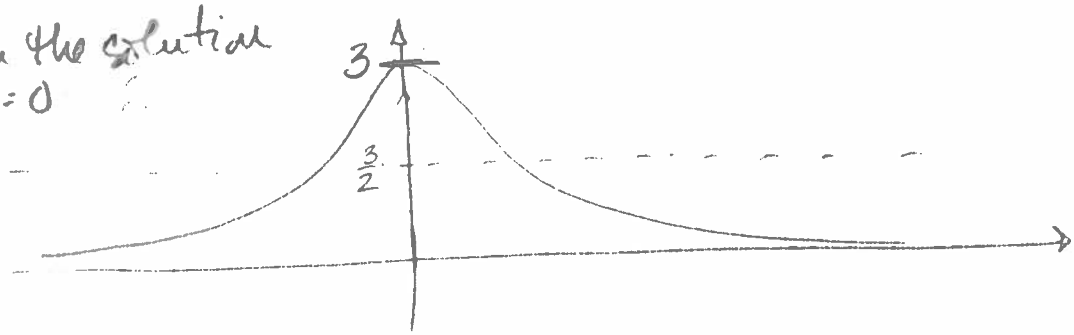
$$u(x,t) = \frac{1}{2} f(x+4t) + \frac{1}{2} f(x-4t).$$

2. $$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \; t > 0$$

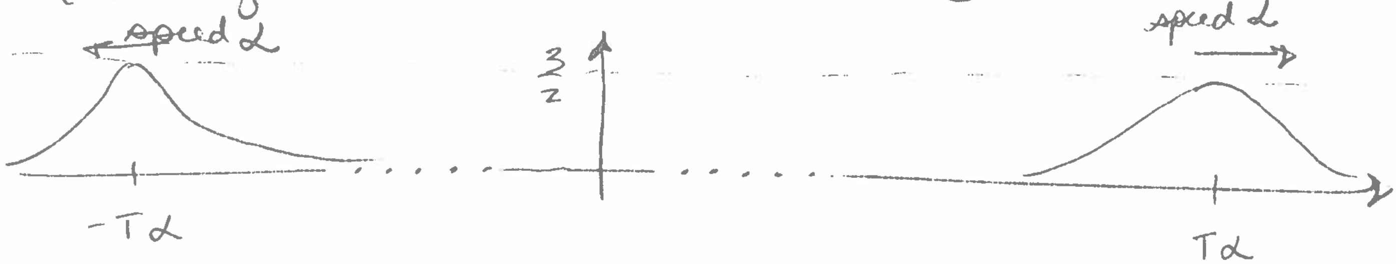
$$u(x,0) = 3e^{-x^2}$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

a) Sketch the solution at $t=0$



b) Sketch the solution at a very large value of $t = T$ (so large that the waves essentially don't overlap).



c) Solve for $u(x,t)$.

Since $\frac{\partial u}{\partial t}(x,0) = 0$, we know that

$$u(x,t) = \frac{1}{2}\left(f(x+\alpha t) + f(x-\alpha t)\right)$$

$$= \frac{1}{2}\left[3e^{-(x+\alpha t)^2} + 3e^{-(x-\alpha t)^2}\right]$$

$$= \frac{3}{2}\left[e^{-(x+\alpha t)^2} + e^{-(x-\alpha t)^2}\right]$$