a place of mind
THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson   Course: MATH 319
Date: Oct 2nd, 2014   Time: 12:30pm Duration: 45 minutes.
This exam has 7 questions for a total of 35 points.

NAME: ______________________ NAME: ______________________

NAME: ______________________ NAME: ______________________

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

This is a two-stage exam. You have 45 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Problem Number	1	2	3	4	5	6	**Total**	BONUS
Points Earned								
Points Out Of	6	3	6	6	6	5	35	5

6 1. Consider the PDE problem below:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,$$
$$u(0,t) = U_1, \quad \frac{\partial u}{\partial x}(L,t) = U_2, \qquad t > 0$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Note that U_1 and U_2 are constants. Derive and solve the steady state problem that arises from the PDE problem above.

3 2. Given that

$$\frac{\partial^2 u}{\partial t^2} = 25 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u_t(x,0) = 0, \qquad u(x,0) = f(x), \qquad -\infty < x < \infty,$$

what is the d'Alembert solution? Write it out algebraically, and then show what this solution means on the axis below by sketching the solution at t=2.

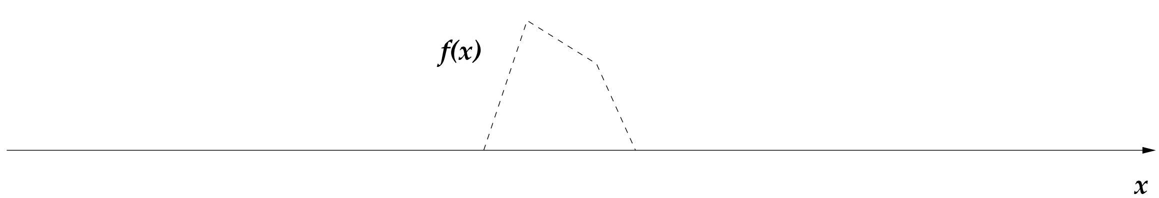

Figure 1: Plot of $f(x)$ for question 2.

6 3. Consider the BVP below:

$$F''(x) + \mu^2 F(x) = 0, \qquad F'(0) = 0, \quad F(1) + F'(1) = 0.$$

Show that this BVP has multiple solutions (assume $\mu > 0$).

4. Laplace's equation on a wedge:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < r < 1, \quad 0 \leq \theta \leq \frac{\pi}{3}, \tag{1}$$

$$u(r,0) = u\left(r, \frac{\pi}{3}\right) = 0, \quad u(1,\theta) = f(\theta), \qquad 0 < r < 1, \quad 0 \leq \theta \leq \frac{\pi}{3}. \tag{2}$$

By assuming $u(r,\theta) = F(r)G(\theta)$, we reduce the PDE problem above to a pair of ODE problems. Solving, we find that the eigenvalues and eigenfunctions of the θ problem are

$$\lambda_n = (3n)^2, \qquad G_n(\theta) = \sin(3n\theta), \quad n \in \mathbb{N} \tag{3}$$

The corresponding problem in r is

$$r^2 F'' + rF' - \lambda F = 0. \tag{4}$$

6 (a) Solve for $F(r)$.

3 (b) Write the formal solution for $u(r,\theta)$ and include the formula for the unknown Fourier coefficients.

6 5. Consider the functions

$$\left\{ \cos\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty} \tag{5}$$

Show that these functions form an orthogonal set on the interval $0 \leq x \leq L$ with respect to the weight function $w(x) = 1$.

5 6. Given

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha(\frac{n\pi}{L})^2 t}, \text{ on } 0 < x < L, \tag{6}$$

where

$$u(x,0) = f(x) = \frac{5}{3}\sin\left(\frac{6\pi x}{L}\right) + 3, \tag{7}$$

find the Fourier coefficients c_n and write out the full solution for $u(x,t)$.

7. BONUS PROBLEM (5 points)

   Consider the following Dirichlet Laplace problem on a rectangular domain:

$$\nabla^2 u = 0, \qquad 0 < x < a, \quad 0 < y < b, \tag{8}$$

$$u(0, y) = u(a, y) = 0, \qquad 0 < y < b, \tag{9}$$

$$u(x, 0) = f_1(x), \quad u(x, b) = f_2(x), \qquad 0 < x < a. \tag{10}$$

   Find the formal solution for $u(x, y)$ (don't solve for the Fourier coefficients).

(Question 7 continued.)

Information That May Be Useful

Note: On this test, you may assume that for the BVP

$$F'' + \lambda F = 0,$$

the only nontrivial solutions are obtained for the case $\lambda = \mu^2 > 0$, and the corresponding solutions are

$$F(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x),$$

without showing any work.

Possibly useful identities:

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$
$$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$