MIDTERM #2 SAMPLE TEST PROBLEMS

COURSE: Differential Equations II (PDEs)
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Disclaimer: This set of sample problems is too long for a midterm test. The actual test would be a subset of problems of the general type that appear here. These problems are provided as a study resource, not as a summary of the course taught so far. Problems on the midterm can come from **any material** in the lectures, tutorials, assignments and pre-reading assignments. These problems however, should give you an idea of the way the questions can be asked so as to be doable in the 45-minute timeframe of the midterm. Note that further shortening of test problems may be necessary, so make sure that you understand each part of the problems below, and could start "in the middle" of any problem given the information leading up to that stage.

Note: I have taken the problems below directly from problems that we solve in class or in tutorial, so you should find all of the answers in your notes. For the midterm, the problems will be a little different.

□ 1. Consider the PDE problem below:

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0, \tag{1}$$

$$u(0,t) = U_1, \quad u(L,t) = U_2, \qquad t > 0 \tag{2}$$

$$u(x,0) = f(x), \qquad 0 < x < L. \tag{3}$$

Note that U_1 and U_2 are constants.

□ (a) Derive and solve the steady state problem that arises from the PDE problem above.

□ (b) Given that the solution to the corresponding homogeneous problem is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right),$$

give the equation for the Fourier coefficients c_n.

□ 2. Consider the BVP below:

$$F''(x) + \lambda F(x) = 0, \qquad F(0) = 0, \quad F(\pi) + F'(\pi) = 0.$$

Show that this BVP has multiple solutions (eigenvalues and eigenfunctions) when $\lambda > 0$.

□ 3. Write out what $\nabla^2 u$ means in (a) cartesian and (b) polar coordinates.

□ 4. Use separation of variables to reduce the PDE problem below to three ODE problems:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad 0 < x < a, \quad 0 < y < b, \quad t > 0, \tag{4}$$

$$u(0,y,t) = u_x(a,y,t) = 0, \qquad 0 < y < b, \quad t > 0 \tag{5}$$

$$u(x,0,t) = u(x,b,t) = 0, \qquad 0 < y < b, \quad t > 0 \tag{6}$$

$$u(x,y,0) = f(x,y), \qquad 0 < x < a, \quad 0 < y < b. \tag{7}$$

☐ 5. Using separation of variables, suppose we find that a formal solution to a certain PDE problem is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{m,n} \sin\left(\frac{(2n-1)\pi x}{2a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-\left[\left(\frac{2n-1}{2a}\right)^2+\left(\frac{m}{b}\right)^2\right]\pi^2 t}.$$

From this solution, deduce the answers to the following questions:

☐ (a) What type of PDE does $u(x,y,t)$ satisfy?

☐ (b) What are the dimensions of the domain on which $u(x,y,t)$ is defined? Draw a sketch.

☐ (c) What were the boundary conditions on u? Add these to your sketch.

☐ 6. Consider the non-homogeneous PDE problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + h(x,t), \qquad 0 < x < L, \quad t > 0, \tag{8}$$

$$u(0,t) = u(L,t) = 0, \qquad t > 0 \tag{9}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < L. \tag{10}$$

We solve this by assuming a series solution for $u(x,t)$.

☐ (a) Why can we not use the assumption $u(x,t) = F(x)G(t)$ to solve this PDE problem?

☐ (b) Assume

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right), \text{ and} \tag{11}$$

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{12}$$

and plug (11) and (12) into the PDE. What is the resulting ODE for $u_n(t)$? How do we solve it?

☐ 7. Consider the PDE problem:

$$\nabla^2 u = 0, \qquad 0 \le r < a, \quad -\pi \le \theta \le \pi, \tag{13}$$

$$u(a,\theta) = f(\theta), \qquad -\pi \le \theta \le \pi. \tag{14}$$

☐ (a) What is the shape of the domain on which this problem is defined?

☐ (b) Determine the two ODE problems that arise from using the separation of variables technique to solve the PDE problem.

☐ 8. Suppose $u(r,\theta) = R(r)T(\theta)$ is governed by a separable PDE, and that the BVP in $T(\theta)$ yields the eigenvalues and eigenfunctions:

$$\text{eigenvalue: } \lambda = 0, \qquad \text{eigenfunction: } B_0, \tag{15}$$

$$\text{eigenvalue: } \lambda = n^2, \quad n = 1, 2, 3 \ldots, \qquad \text{eigenfunction: } A_n\cos(n\theta) + B_n \sin(n\theta). \tag{16}$$

The ODE in $R(r)$ is

$$r^2 R'' + rR' - \lambda R = 0. \tag{17}$$

Solve (17) for both cases of the eigenvalue λ.

☐ 9. Consider the PDE problem

$$\frac{\partial^2 u}{\partial t^2} = 4^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0, \tag{18}$$

$$u_t(x,0) = 0, \qquad -\infty < x < \infty, \tag{19}$$

$$u(x,0) = \begin{cases} 2(x^2-1) & \text{if } |x| \le 1, \\ 0 & \text{else} \end{cases} \tag{20}$$

☐ (a) Sketch the solution at $t = 0$ and 3..

☐ (b) Compute the solution analytically.

☐ 10. Consider the functions

$$\left\{ \sin \frac{n\pi x}{L} \right\}_{n=1}^{\infty} \tag{21}$$

(i). Show that these functions fom an orthogonal set on the interval $0 \le x \le L$ with respect to the weight function $w(x) = 1$.

(ii). Given

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp^{-\alpha(\frac{n\pi}{L})^2 t}, \tag{22}$$

where $u(x,0) = f(x) = 2\sin\left(\frac{3\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{7\pi x}{L}\right) - 5$, use your result from part (i) to help find the Fourier coefficients c_n and write out the full solution for $u(x,t)$.

☐ 11. Consider the Dirichlet problem on a rectangle:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b, \tag{23}$$

$$u(0,y) = u(a,y) = 0, \qquad 0 < y < b \tag{24}$$

$$u(x,0) = f(x), \quad u_y(x,b) = g(x), \qquad 0 < x < a. \tag{25}$$

Suppose that $u(x,y) = F(x)G(y)$. The BVP in $F(x)$ yields the eigenvalues and eigenfunctions:

$$\text{Eigenvalues:} \qquad \lambda_n = \left(\frac{n\pi}{a}\right)^2 \text{ for } n \in \mathbb{N} \tag{26}$$

$$\text{Eigenfunctions:} \qquad F_n(x) = c_n \sin\left(\frac{n\pi x}{a}\right). \tag{27}$$

Find the formal solution for $u(x,t)$. (*HINT: Use the trick that you learned in tutorial to pick a general form for $G(y)$ that will make it easier to solve for the coefficients in your Fourier series when you apply the conditions in (25).*)