1)

Math 319 - Assignment #1

1) Find the general solution of the given second-order differential equation:

a) $4y''+y'=0$.

b) $y''-y'-6y=0$

c) $12y''-5y'-2y=0$.

Solution:

a) $4y''+y'=0$

Let $y=e^{rt}$

$y'=re^{rt}$

$y''=r^2e^{rt}$

$\therefore 4r^2e^{rt}+re^{rt}=0$

$(4r^2+r)e^{rt}=0$

$4r^2+r=0$

$r(4r+1)=0$

$r=0 \quad r=-\frac{1}{4}$

$\therefore \boxed{y(t)=c_1+c_2e^{-\frac{1}{4}t}}$

①

b) $y''-y'-6y=0$

Let $y=e^{rt}$

$y'=re^{rt}$

$y''=r^2e^{rt}$

$\therefore r^2e^{rt}-re^{rt}-6e^{rt}=0$

$(r^2-r-6)e^{rt}=0$

$r^2-r-6=0$

$(r-3)(r+2)=0$

$r=3 \quad r=-2$

$\therefore \boxed{y(t)=c_1e^{3t}+c_2e^{-2t}}$

①

c) $12y''-5y'-2y=0$

Let $y=e^{rt}$

$y'=re^{rt}$

$y''=r^2e^{rt}$

$\therefore 12r^2e^{rt}-5re^{rt}-2e^{rt}=$

$(12r^2-5r-2)e^{rt}=0$

$12r^2-5r-2=0$

$M=-24 \quad -8,3$

$A=-5$

$12r^2-8r+3r-2=0$

$4r(3r-2)+1(3r-2)=0$

$(4r+1)(3r-2)=0$

$r=-\frac{1}{4} \quad r=\frac{2}{3}$

$\therefore \boxed{y(t)=c_1e^{-\frac{1}{4}t}+c_2e^{\frac{2}{3}t}}$

①

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2) Solve the initial value problem

$y''+y'+2y=0$

$y(0)=1,\ y'(0)=0.$

Solution:

Let $y=e^{rt}$

$y'=re^{rt}$

$y''=r^2e^{rt}$

$y''+y'+2y=0$

$r^2e^{rt}+re^{rt}+2e^{rt}=0$

$(r^2+r+2)e^{rt}=0$

①

$r^2+r+2=0$

$r=\frac{-1\pm\sqrt{1^2-4(1)(2)}}{2(1)}=\frac{-1\pm\sqrt{-7}}{2}=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

(2)

$\therefore y(t) = e^{-\frac{1}{2}t}\left[c_1\cos\left(\frac{\sqrt{7}}{2}t\right) + c_2\sin\left(\frac{\sqrt{7}}{2}t\right)\right]$ (0.5)

Apply the ICs

$y(0) = 1$

$y(0) = c_1 = 1$ (0.5)

$\Rightarrow y(t) = e^{-\frac{1}{2}t}\left[\cos\left(\frac{\sqrt{7}}{2}t\right) + c_2\sin\left(\frac{\sqrt{7}}{2}t\right)\right]$

$y'(t) = -\frac{1}{2}e^{-\frac{1}{2}t}\left[\cos\left(\frac{\sqrt{7}}{2}t\right) + c_2\sin\left(\frac{\sqrt{7}}{2}t\right)\right] + e^{-\frac{1}{2}t}\left[-\frac{\sqrt{7}}{2}\sin\left(\frac{\sqrt{7}}{2}t\right) + c_2\frac{\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}\right.\right.$

$y'(0) = -\frac{1}{2}(1)\left[(1)+0\right] + (1)\left[0 + c_2\frac{\sqrt{7}}{2}(1)\right] = 0$

$-\frac{1}{2} + \frac{\sqrt{7}}{2}c_2 = 0$

$\frac{\sqrt{7}}{2}c_2 = \frac{1}{2}$

$c_2 = \frac{1}{\sqrt{7}}$ (0.5)

$\therefore \boxed{y(t) = e^{-\frac{1}{2}t}\left[\cos\left(\frac{\sqrt{7}}{2}t\right) + \frac{1}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right]}$ (0.5)

/3

3) Consider the differential equation $y'' + \lambda y = 0$, with bundary conditions $y(0) = 0$ and $y(\pi/2) = 0$. This is called a bundary value problem, because instead of conditions on y & y' at a single time (say 0 or $\pi/2$), they are given both and at different times (0 and $\pi/2$). Is it possible to determine values of λ so that the problem possesses

(a) only trivial solutions?

(b) some nontrivial solutions?

Solution:

Let $y = e^{rt}$
$y' = re^{rt}$
$y'' = r^2e^{rt}$

$y'' + \lambda y = 0$
$r^2e^{rt} + \lambda e^{rt} = 0$
$(r^2 + \lambda)e^{rt} = 0$
$r^2 = -\lambda$
$r = \pm\sqrt{-\lambda}$.

Case 1: $\lambda = 0$

$y'' = 0$
$y' = A$
$y = At + B$

$y(0) = B = 0$
$y(\pi/2) = A(\pi/2) = 0$
$\Rightarrow A = 0$

Trivial Solution. (1)

Case 2: $\lambda < 0$, so set $\lambda = -\mu^2$ with $\mu > 0$

Then $r = \pm\mu$

$y(t) = c_1 \cosh \mu t + c_2 \sinh \mu t$

$y(0) = c_1(1) = 0 \Rightarrow c_1 = 0$ (1)

$y(\pi/2) = \underbrace{c_2 \sinh \mu \pi/2}_{\neq 0 \text{ since } \mu > 0} = 0$

$\therefore c_2 = 0$. *Trivial Solution*.

Case 3: $\lambda > 0$, so set $\lambda = \mu^2$ with $\mu > 0$.

Then $r = \pm i\mu$.

$y(t) = c_1 \cos \mu t + c_2 \sin \mu t$

$y(0) = c_1 = 0$ (1)

$y(\pi/2) = c_2 \sin \mu \pi/2 = 0$

$c_2 = 0$ Trivial Solution

$\sin \mu \pi/2 = 0$

$\mu \frac{\cancel{\pi}}{2} = n\cancel{\pi}$ for $n = 1, 2, \ldots$

$\mu = 2n$ for $n = 1, 2, \ldots$

a) Only trivial solutions are possible if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$ but $\lambda = \mu^2 \neq 4n^2$ for $n = 1, 2, \ldots$ ~~2~~

b) Nontrivial Solutions are obtained if $\lambda = \mu^2 = 4n^2$ for $n = 1, 2, \ldots$

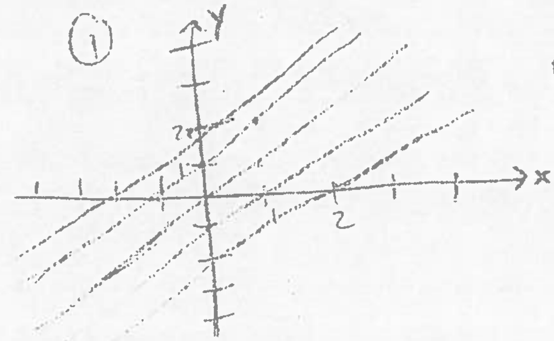
$\boxed{/3}$

4) Determine and graph the family of characteristic lines for the PDE $u_x(x,t) + u_t(x,t) = 0$

Solution: $a = 1$, $b = 1$, slope $= 1$

characteristic lines: $bx - ay = d$ (1) $\boxed{/2}$

$x - y = d$ (1)



④

5. Find the general solution of $u_x + u_y + u = e^{-3y}$.

We use the method of characteristic lines.

Since

$$u_x + u_y + u = e^{-3y}, \qquad (1)$$

the characteristic lines have slope 1. We set $v(w,z) = u(x,y)$, where

$$\begin{cases} w = x - y \\ z = y \end{cases} \quad \Rightarrow \quad \begin{cases} x = w + z \\ y = z \end{cases}$$

This change of variables yields

$$\begin{aligned} u_x + u_y =& (v_w w_x + v_z z_x) + (v_w w_y + v_z z_y) \\ =& v_w - v_w + v_z \\ =& v_z. \end{aligned}$$

Substitution in (1) gives

$$v_z + v = e^{-3z}.$$

We multiply both sides by the integrating factor, e^z, and combine the terms to get

$$(e^z v)_z = e^{-2z}.$$

We integrate both sides of the equation with respect to z:

$$e^z v = \int e^{-2z}\, dz = -\tfrac{1}{2} e^{-2z} + c(w).$$

So $v(w,z) = -\frac{e^{-3z}}{2} + e^{-z} c(w)$, where $c(w)$ is an arbitrary function of w. The general solution in terms of x and y is given by

$$u(x,y) = -\frac{e^{-3y}}{2} + e^{-y} F(x-y),$$

where F is an arbitrary function of $x - y$.

5

a) Consider the hyperbolic PDE (also called the "wave equation" or "Vibrating string equation")

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

with BCs and initial values given by

$$u(0,t) = u(4,t) = 0, \qquad u(x,0) = 4\sin\left(\frac{3\pi x}{4}\right) \qquad (2)$$

i) Show that

$$u(x,t) = K\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi t}{4}\right) \qquad (3)$$

is a solution to (1) with (2). What must be the value of K?

ii) Sketch the solution at t=0 and at t=0.5.

Solution:

a) $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$

$$K\sin\left(\frac{3\pi x}{4}\right)\left(\frac{6\pi}{4}\right)^2\left(-\cos\left(\frac{6\pi t}{4}\right)\right) \stackrel{?}{=} 4K\left(\frac{3\pi}{4}\right)^2 \cdot -\sin\left(\frac{3\pi}{4}x\right)\cos\left(\frac{6\pi t}{4}\right) \quad (1)$$

$$-\frac{36\pi^2}{16}K\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi t}{4}\right) = \frac{-36\pi^2}{16}K\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi t}{4}\right)$$

$\therefore u(x,t) = K\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi t}{4}\right)$ is a solution to (1)

$$u(0,t) = K(0)\cos\left(\frac{6\pi t}{4}\right) = 0$$

$$u(4,t) = K\underbrace{\sin\left(\frac{3\pi\cdot 4}{4}\right)}_{\sin 3\pi = 0}\cos\left(\frac{6\pi t}{4}\right) = 0 \qquad (0.5)$$

4

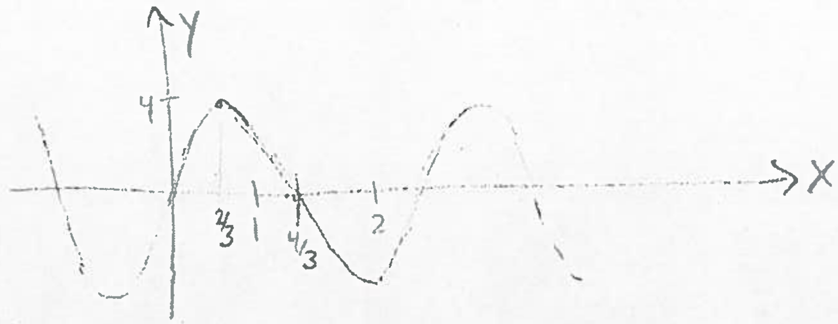
$$u(x,0) = K\sin\left(\frac{3\pi x}{4}\right)(1) = 4\sin\left(\frac{3\pi x}{4}\right) \qquad (0.5)$$

$$\Rightarrow \boxed{K=4}$$

b) $t=0$

$$u(x,0)=4\sin\left(\frac{3\pi x}{4}\right)$$

x	u
0	0
2/3	4
4/3	0
6/3=2	-4



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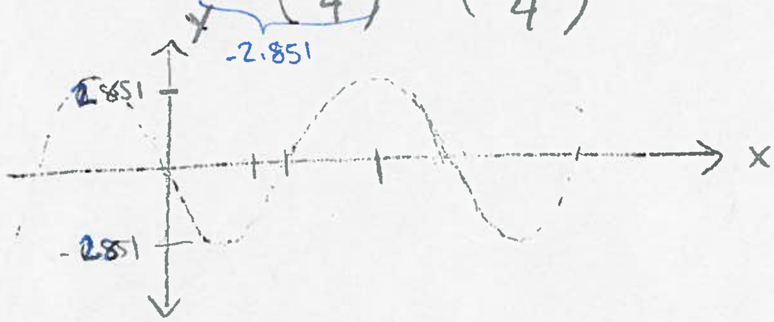
$t=0.5$

$$u(x,0.5)=4\sin\left(\frac{3\pi x}{4}\right)\cos\left(\frac{6\pi\cdot 0.5}{4}\right)$$

$$=\underbrace{4\cos\left(\frac{3\pi}{4}\right)}_{-2.851}\sin\left(\frac{3\pi x}{4}\right)$$



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1) By inspection, determine the coefficients b_n in the furier series

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x)=5\sin(2\pi x)-7\sin(5\pi x)+2\sin(9\pi x)$$

Solution:

$b_2=5$

$b_5=-7$

$b_9=2$

$b_n=0 \quad \forall\, n\neq 2,5,9.$

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