# Math 319 - Differential Equations II
# Assignment # 2
# due Mon Oct 3rd, 11:30am, SCI 386

**Instructions:** You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

The assignment may be done with up to 4 other classmates (i.e. total group size: no more than 5). If you collaborate with classmates, the group should hand in one document with all contributing names at the top.

1. Use the method of separation of variables to derive the set of ODEs associated with the following partial differential equation for $u = u(x,t)$:
$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

2. Use separation of variables to solve the heat flow problem
$$\begin{aligned} &\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad t > 0,\\ &u(0,t) = u(\pi,t) = 0, && t > 0,\\ &u(x,0) = \sin(x) - 7\sin(3x) + \sin(5x), && 0 < x < \pi. \end{aligned}$$

3. The Legendre polynomials, $P_n(x)$, are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(x) = 1$. Use the fact that
$$P_0(x) = 1, \quad P_1(x) = x, \quad \text{and} \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
to find the first three coefficients in the expansion $f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + ...$, where $f(x)$ is the function given by
$$f(x) = \begin{cases} -1, & -1 < x < 0,\\ 1, & 0 < x < 1. \end{cases}$$

4. Obtain (a) the Fourier sine series, and (b) the Fourier cosine series for the function
$$f(x) = x + \pi \quad \text{on} \quad 0 < x < \pi.$$
For each series, discuss the convergence and show graphically (using Maple) the function represented by the series for all x (your graph should show three periods of the function).

5. Consider the function
$$f(x) = \begin{cases} |x|, & \text{on} \quad -\pi \le x \le \pi\\ f(x+2\pi), & \text{else.} \end{cases}$$
Use Maple to investigate the speed with which the Fourier series for $f$ converges. In particular, determine how many terms are needed so that the error is no greater than 0.05 for all $x$ in the interval $[-1,1]$. Your solution should include plots of $f(x)$ and the truncated Fourier series $s_m(x)$, as well as plots of the error $e_m(x) = |f(x) - s_m(x)|$ for various values of $m$.

# Math 319, A#2, Sol'ns

Sept 18th

Lindsey Reinholz

1. Use the method of separation of variables to derive the set of ODEs associated with the following partial differential equation for $u = u(x,t)$.

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Solution:

By the method of separation of variables, the solution should have the form
$u(x,t) = X(x)T(t)$.

$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$ $\quad$ ($X(x)$ is constant wrt $t$)

$\frac{\partial u}{\partial t} = X(x)T'(t)$ $\quad$ ($X(x)$ is constant wrt $t$)

$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ $\quad$ ($T(t)$ is constant wrt $x$)

Substituting into the PDE yields

$$X(x)T''(t) + X(x)T'(t) + X(x)T(t) = \alpha^2 X''(x)T(t)$$

Dividing by $\alpha^2 X(x)T(t)$ gives

$$\underbrace{\frac{T''(t)}{\alpha^2 T(t)} + \frac{T'(t)}{\alpha^2 T(t)} + \frac{1}{\alpha^2}}_{\text{function of } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x}$$

$$\frac{1}{\alpha^2}\left(\frac{T''(t)}{T(t)} + \frac{T'(t)}{T(t)} + 1\right) = \frac{X''(x)}{X(x)} = -\lambda \qquad \lambda = \text{constant}$$

1. $\frac{X''(x)}{X(x)} = -\lambda$

$X''(x) = -\lambda X(x)$

$$\boxed{X''(x) + \lambda X(x) = 0}$$

② $\frac{1}{\alpha^2}\left(\frac{T''(t)}{T(t)} + \frac{T'(t)}{T(t)} + 1\right) = -\lambda$

$$T''(t) + T'(t) + T(t) = -\lambda\alpha^2 T(t)$$

$$\boxed{T''(t) + T'(t) + (1+\lambda\alpha^2)T(t) = 0}$$

Thus, the solution $u(x,t) = X(x)T(t)$ satisfies the ordinary differential equations

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + T'(t) + (1+\lambda\alpha^2)T(t) = 0 .$$

2. $$\begin{cases} \dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2} & 0<x<\pi,\ t>0 \\ u(0,t)=u(\pi,t)=0 & t>0 \\ u(x,0)=\sin(x)-7\sin(3x)+\sin(5x) & 0<x<\pi \end{cases}$$

Steps 1 + 2: Separate + form ODEs

Let $u(x,t)=X(x)T(t)$. Then the PDE becomes

$$XT' = 3X''T \Leftrightarrow \frac{T'}{3T} = \frac{X''}{X} = -\lambda$$

where $\lambda$ is a constant. We thus arrive at two ODEs:

(A) $X''+\lambda X=0, \quad X(0)=X(\pi)=0$

(B) $T'+3\lambda T=0$

Step 3: Solve the ODEs

(A) case 1: $\lambda=-\omega^2<0$

Then $X(x)=c_1 e^{\omega x}+c_2 e^{-\omega x}$.

BCs: $\begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases} \Leftrightarrow \begin{cases} c_1+c_2=0 \\ c_1 e^{\omega\pi}+c_2 e^{-\omega\pi}=0 \end{cases} \Leftrightarrow \begin{cases} c_1=-c_2 \\ c_2\left(e^{\omega\pi}-e^{-\omega\pi}\right)=( \end{cases}$

$\therefore$ either $\omega = 0$ or $c_1 = -c_2 = 0$, &
so we only have trivial solutions
in this case.

<u>case 2: $\lambda = 0$</u>

Then $X(x) = c_1 x + c_2$.

$$\text{BCs: } \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \Leftrightarrow \begin{cases} c_2 = 0 \\ c_1 \pi = 0 \end{cases} \Leftrightarrow \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases}$$

$\therefore$ we only have trivial solutions
in this case.

<u>case 3: $\lambda = \omega^2 > 0$</u>

Then $X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$.

$$\text{BCs: } \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 \sin(\omega \pi) = 0 \end{cases}$$

$\therefore$ for nontrivial solutions we
require

$$\omega \pi = n\pi \Leftrightarrow \omega_n = n$$

and the eigenfunctions are

$$X_n(x) = c_n \sin(nx).$$

(B) $T' + 3\lambda T = 0 \iff T' + 3\omega^2 T = 0$

For each value of $\omega$ we have

$$T_n' + 3n^2 T = 0 \iff T_n = d_n e^{-3n^2 t}$$

Step 4: Superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\, d_n e^{-3n^2 t}$$

$$= \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-3n^2 t}$$

Step 5: Apply ICs:

$$u(x,0) = \sin(x) - 7\sin(3x) + \sin(5x)$$

$$= \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\therefore \begin{cases} b_1 = 1,\ b_3 = -7,\ b_5 = 1 \\ b_n = 0 \ \forall n \in \mathbb{N} \neq 1,3,5 \end{cases}$$

The final solution is

$$u(x,t) = \left[\sin(x)\, e^{-3t} - 7\sin(3x)\, e^{-27t} + \sin(5x)\, e^{-75t}\right]$$

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# Math 319 - Assignment #2

3. The Legendre polynomials $P_m(x)$ and $P_n(x)$ are said to be orthogonal in the interval $-1 \leq x \leq 1$ provided

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad m \neq n$$

and as a result, we have

$$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{n+1} \qquad m = n.$$

Any function $f(x)$ which is finite and single-valued in the interval $-1 \leq x \leq 1$, and which has a finite number of discontinuities within this interval can be expressed as a series of Legendre polynomials.

We let

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots \qquad -1 \leq x \leq 1$$

$$= \sum_{n=0}^{\infty} a_n P_n(x).$$

Multiplying both sides by $P_m(x)\,dx$ and integrating with respect to $x$ from $x=-1$ to $x=1$ gives

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\, dx$$

By means of the orthogonality property of the Legendre polynomials we can write

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \qquad n = 0, 1, 2, 3, \ldots$$

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$.

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots$$

To find the first coefficient, $a_0$, multiply both sides by $P_0(x)dx$ and integrate wrt $x$ from $x=-1$ to $x=1$.

$$\int_{-1}^{1} f(x) P_0(x) dx = \int_{-1}^{1} (a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots) P_0(x) dx$$

$$= a_0 \int_{-1}^{1} P_0^2(x) dx + a_1 \underbrace{\int_{-1}^{1} P_0(x) P_1(x) dx}_{\to 0} + a_2 \underbrace{\int_{-1}^{1} P_0(x) P_2(x) dx}_{\to 0} + \cdots$$

By orthogonality condition
(all the rest equal 0 too)

$$\int_{-1}^{1} f(x) P_0(x) dx = a_0 \int_{-1}^{1} (1)^2 dx$$

$$= a_0 \, x \Big|_{-1}^{1}$$

$$= a_0 (1 - (-1))$$

$$= 2a_0$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x) P_0(x) dx$$

$$= \frac{1}{2} \left[ \int_{-1}^{0} -1 (1) dx + \int_{0}^{1} (1)(1) dx \right]$$

$$= \frac{1}{2} \left[ -x \Big|_{-1}^{0} + x \Big|_{0}^{1} \right]$$

$$= \frac{1}{2} \left[ 0 - (-(-1)) + 1 - 0 \right]$$

$$= \frac{1}{2} [-1 + 1] = 0.$$

$$\boxed{a_0 = 0}$$

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To find the second coefficient, $a_1$, multiply both sides by $P_1(x)dx$ and integrate wrt $x$ from $x=-1$ to $x=1$.

$$\int_{-1}^{1} f(x)P_1(x)dx = \int_{-1}^{1}(a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+\cdots)P_1(x)dx$$

$$= a_0\cancelto{0}{\int_{-1}^{1}P_0(x)P_1(x)dx} + a_1\int_{-1}^{1}P_1^2(x)dx + a_2\cancelto{0}{\int_{-1}^{1}P_1(x)P_2(x)dx} + \cdots = 0$$

By orthogonality condition

$$= a_1\int_{-1}^{1}P_1^2(x)dx$$

$$= a_1\int_{-1}^{1}x^2dx$$

$$= a_1 \left.\frac{x^3}{3}\right|_{-1}^{1}$$

$$= a_1\left(\frac{1}{3}-\frac{(-1)^3}{3}\right)$$

$$= \frac{2}{3}a_1$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}f(x)P_1(x)dx$$

$$= \frac{3}{2}\left[\int_{-1}^{0}(-1)x\,dx + \int_{0}^{1}(1)x\,dx\right]$$

$$= \frac{3}{2}\left[\left.\frac{-x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{1}\right]$$

$$= \frac{3}{2}\left[0-\left(-\frac{(-1)^2}{2}\right)+\frac{1^2}{2}-0\right]$$

$$= \frac{3}{2}\left[\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{3}{2}$$

$$\boxed{a_1 = \frac{3}{2}}$$

To find the third coefficient, $a_2$, multiply both sides by $P_2(x)dx$ and integrate wrt $x$ from $x=-1$ to $x=1$

$$\int_{-1}^{1} f(x)P_2(x)\,dx = \int_{-1}^{1}\left(a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+\cdots\right)P_2(x)\,dx$$

$$= a_0\underbrace{\int_{-1}^{1}P_0(x)P_2(x)\,dx}_{\to 0} + a_1\underbrace{\int_{-1}^{1}P_1(x)P_2(x)\,dx}_{\to 0} + a_2\int_{-1}^{1}P_2^2(x)\,dx + \underbrace{\cdots}_{=0}$$

By orthogonality condition

$$= a_2\int_{-1}^{1}P_2^2(x)\,dx$$

$$= a_2\int_{-1}^{1}\left(\frac{3}{2}x^2-\frac{1}{2}\right)^2 dx$$

$$= a_2\int_{-1}^{1}\left(\frac{9}{4}x^4-\frac{3}{2}x^2+\frac{1}{4}\right)dx$$

$$= a_2\left[\frac{9}{4}\frac{x^5}{5}-\frac{3}{2}\frac{x^3}{3}+\frac{1}{4}x\right]\Bigg|_{-1}^{1}$$

$$= a_2\left[\frac{9}{4}\cdot\frac{(1)^5}{5}-\frac{3}{2}\cdot\frac{(1)^3}{3}+\frac{1}{4}(1)-\left(\frac{9}{4}\frac{(-1)^5}{5}-\frac{3}{2}\cdot\frac{(-1)^3}{3}+\frac{1}{4}(-1)\right)\right]$$

$$= a_2\left[\frac{9}{20}-\frac{1}{2}+\frac{1}{4}+\frac{9}{20}-\frac{1}{2}+\frac{1}{4}\right]$$

$$= a_2\left[\frac{9}{20}-\frac{10}{20}+\frac{5}{20}+\frac{9}{20}-\frac{10}{20}+\frac{5}{20}\right]$$

$$= a_2\left[\frac{8}{20}\right]$$

$$= a_2\left[\frac{2}{5}\right]$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\,dx$$

$$= \frac{5}{2}\left[\int_{-1}^{0}(-1)\left(\frac{3}{2}x^2-\frac{1}{2}\right)dx + \int_{0}^{1}(1)\left(\frac{3}{2}x^2-\frac{1}{2}\right)dx\right]$$

$$= \frac{5}{2}\left[(-1)\left(\frac{\cancel{3}}{2}\frac{x^3}{\cancel{3}}-\frac{1}{2}x\right)\Bigg|_{-1}^{0} + \left(\frac{\cancel{3}}{2}\cdot\frac{x^3}{\cancel{3}}-\frac{1}{2}x\right)\Bigg|_{0}^{1}\right]$$

$$= \frac{5}{2}\left[\frac{1}{2}\left(0-0-\left((-1)^3-(-1)\right)\right)+\frac{1}{2}\left((1-1)-(0-0)\right)\right]$$

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$$a_2 = \frac{5}{2}\left[\frac{1}{2}(0) + \frac{1}{2}(0)\right] = 0.$$

$$\boxed{a_2 = 0}$$

Thus, the first three coefficients in the expansion

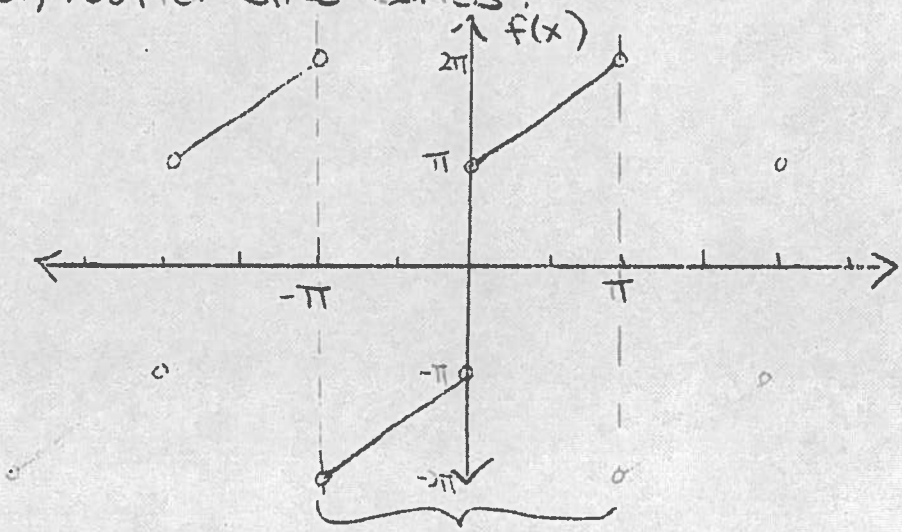
$$f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \cdots$$

are $a_0 = 0$, $a_1 = 3/2$ and $a_2 = 0$, so

$$f(x) = \frac{3}{2}P_1(x) + \cdots$$

4. $f(x) = x + \pi$ on $0 < x < \pi$

a) Fourier Sine Series

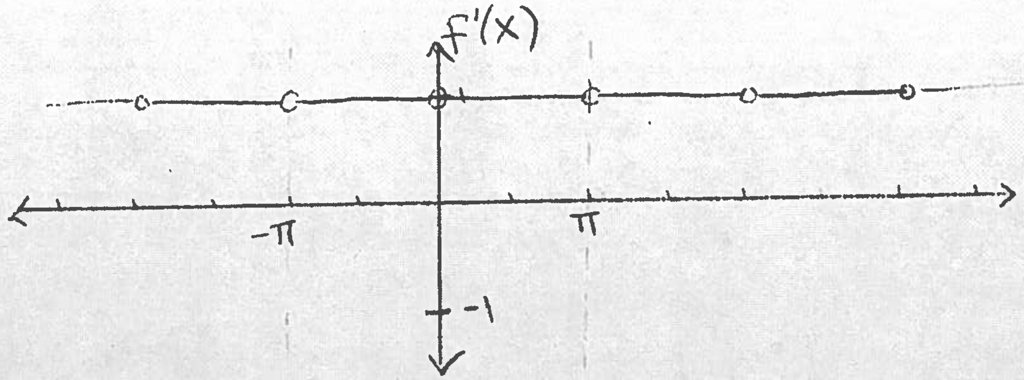



Sine series
↳ Odd Extension

period $= 2\pi \Longrightarrow L = \pi$

$$f(x) = \begin{cases} x - \pi, & -\pi < x < 0 \\ x + \pi, & 0 < x < \pi \end{cases} \qquad f'(x) = \begin{cases} 1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

$f(x)$ is piece-wise continuous on $[-\pi, \pi]$.




$f'(x)$ is piece-wise continuous on $[-\pi, \pi]$.

Since both $f$ and $f'$ are piece-wise continuous on $[-\pi,\pi]$, then for every $x \in (-\pi, \pi)$, the series converges pointwise to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

For $x = \pm L$, the series converges to

$$\frac{1}{2}\left[f(-L^+) + f(L^-)\right] \quad \text{where } L = \pi$$

At the point $x = 0$, the series will converge to

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi + (-\pi)}{2} = 0.$$

At the endpoints, the series will converge to

$$\frac{f(-\pi^+) + f(\pi^-)}{2} = \frac{-2\pi + 2\pi}{2} = 0.$$

Thus, the Fourier Series will converge pointwise to:

$$f(x) = \begin{cases} 0 & x = -\pi \\ x - \pi & -\pi < x < 0 \\ 0 & x = 0 \\ x + \pi & 0 < x < \pi \\ 0 & x = \pi \end{cases}$$

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The Fourier Series for f(x) will have the form:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \qquad n = 1, 2, 3, \ldots$$

$$= \frac{2}{\pi}\int_0^{\pi} (x+\pi) \sin \frac{n\pi x}{\pi}\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} x \sin nx\, dx + \int_0^{\pi} \pi \sin nx\, dx\right]$$

Integration by Parts:

$u = x \qquad dv = \sin nx\, dx$

$du = dx \qquad v = \frac{-1}{n}\cos nx$

$$= \frac{2}{\pi}\left[\frac{-x}{n}\cos nx\Big|_0^{\pi} - \int_0^{\pi} \frac{-1}{n}\cos nx\, dx - \frac{\pi}{n}\cos nx\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n^2}\sin nx\Big|_0^{\pi} - \frac{\pi}{n}\cos nx\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - \frac{\pi}{n}\cos n\pi - \left(0 + 0 - \frac{\pi}{n}\cos(0)\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n - \frac{\pi}{n}(-1)^n + \frac{\pi}{n}\right]$$

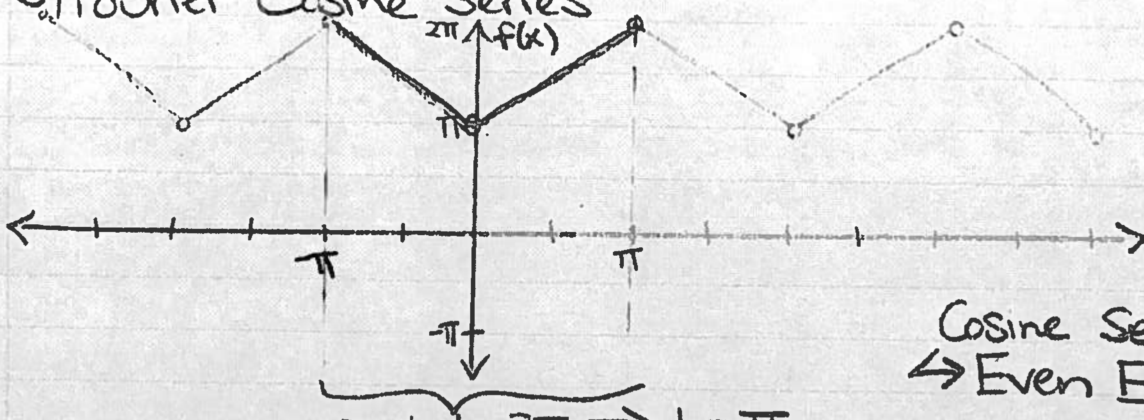
$$= \frac{2}{n}\left(-2(-1)^n + 1\right)$$

$$\boxed{b_n = \frac{2}{n}\left(2(-1)^{n+1} + 1\right)}$$

The Fourier Sine Series is:

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n}\left(2(-1)^{n+1} + 1\right) \sin nx}$$

b) Fourier Cosine Series

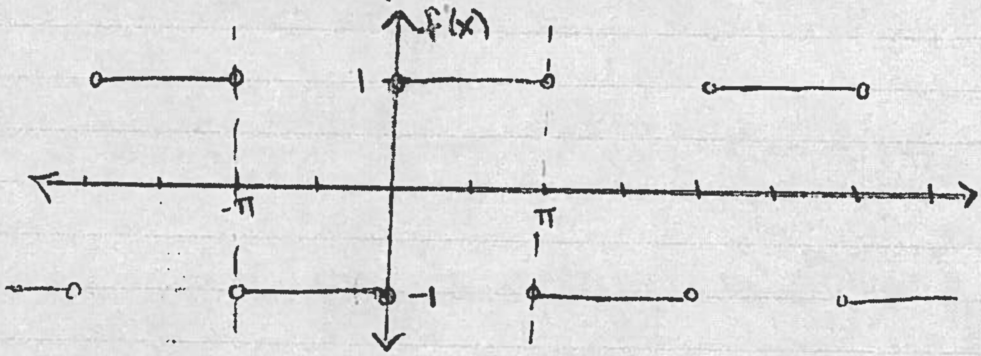


Cosine Series
$\leftrightarrow$ Even Extension



$$f(x) = \begin{cases} -x+\pi & -\pi < x < 0 \\ x+\pi, & 0 < x < \pi \end{cases} \qquad f'(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$f(x)$ is continuous on $[-\pi, \pi]$.



$f'(x)$ is piece-wise continuous on $[-\pi, \pi]$.

Since $f$ is continuous and $f'$ is piece-wise continuous on $[-\pi, \pi]$ then the series converges uniformly to $f$ on $[-\pi, \pi]$.

The Fourier Cosine Series for $f(x)$ will have the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

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$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi (x+\pi)\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}+\pi x\right]\Big|_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}+\pi\cdot\pi - 0\right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{3\pi^{\cancel{2}}}{\cancel{2}}\right]$$

$$= 3\pi$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi (x+\pi)\cos\frac{n\pi x}{\pi}\,dx$$

$$= \frac{2}{\pi}\left[\int_0^\pi x\cos nx\,dx + \int_0^\pi \pi\cos nx\,dx\right]$$

Integration by Parts:

$u = x \qquad dv = \cos nx\,dx$

$du = dx \qquad v = \frac{1}{n}\sin nx$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^\pi - \int_0^\pi \frac{1}{n}\sin nx\,dx + \frac{\pi}{n}\sin nx\Big|_0^\pi\right]$$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^\pi + \frac{1}{n^2}\cos nx\Big|_0^\pi + \frac{\pi}{n}\sin nx\Big|_0^\pi\right]$$

$$= \frac{2}{\pi}\left[\overset{0}{\cancel{\frac{\pi}{n}\sin n\pi}} + \frac{1}{n^2}\cos n\pi + \overset{0}{\cancel{\frac{\pi}{n}\sin n\pi}} - \left(0+\frac{1}{n^2}\cos(0)+0\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right).$$

Note that if

n = even $\Longrightarrow \frac{2}{n^2\pi}\left((-1)^n - 1\right) = 0$

n = odd $\Longrightarrow \frac{2}{n^2\pi}\left((-1)^n - 1\right) = \frac{-4}{n^2\pi}$

Thus, the Fourier Cosine Series is:

$$\boxed{f(x) = \frac{3\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)^2\pi}\cos(2n-1)x}$$

5. $h(x) = |x|$

$$f(x) = \begin{cases} h(x), & \text{on } -\pi < x < \pi \\ h(x+2n\pi), & \text{on } -(1+2n)\pi < x < (1-2n)\pi, \\ & n = \pm 1, \pm 2, \ldots \end{cases}$$

$\because$ $h(x)$ is even, the Fourier series for $f(x)$ will be a Fourier cosine series:

$$a_0 = \frac{1}{\pi}\int_{\pi}^{\pi} h(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{1}{\pi}(\pi^2 - 0)$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\frac{1}{n}\left[\overbrace{\pi\sin(n\pi)}^{0} + \frac{1}{n}\cos(n\pi) - 0 - \frac{1}{n}\right]$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos(nx)$$

**Assisgnment #2, Problem #5, Maple portion**

First I define the periodic function *f(x),* and its Fourier series.

> $f := x \rightarrow piecewise(-\mathrm{Pi} < x \textbf{ and } x < \mathrm{Pi}, |x|, -3\cdot\mathrm{Pi} < x \textbf{ and } x < -\mathrm{Pi}, |x+2\cdot\mathrm{Pi}|, \mathrm{Pi} < x \textbf{ and } x < 3\cdot\mathrm{Pi}, |x-2\cdot\mathrm{Pi}|);$

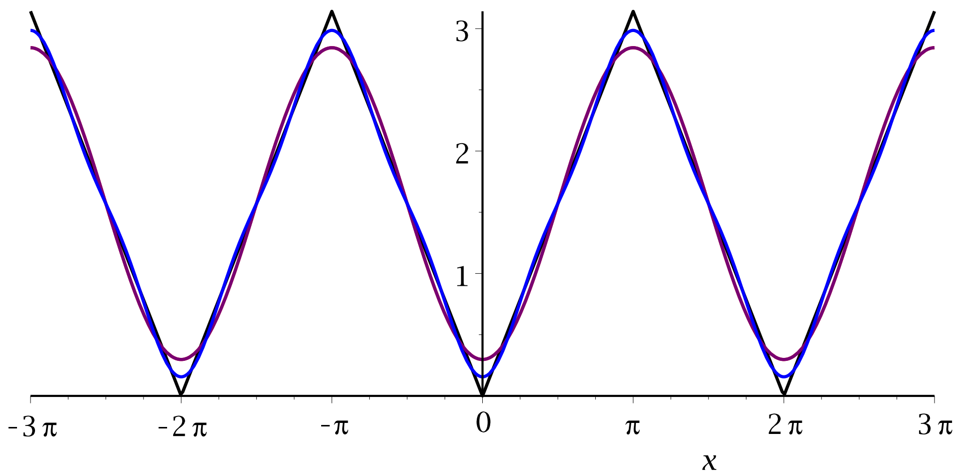
$$f := x \rightarrow piecewise(-\pi < x \textbf{ and } x < \pi, |x|, -3\pi < x \textbf{ and } x < -\pi, |x+2\pi|, \pi < x \textbf{ and } x < 3\pi, |x-2\pi|) \quad \textbf{(1)}$$

> $Ff := (x, nmax) \rightarrow \frac{\mathrm{Pi}}{2} + sum\left(\frac{2\cdot((-1)^n - 1)}{\mathrm{Pi}\cdot n^2}\cdot\cos(n\cdot x), n = 1..nmax\right);$

$$Ff := (x, nmax) \rightarrow \frac{1}{2}\pi + \sum_{n=1}^{nmax} \frac{(2(-1)^n - 2)\cos(n x)}{\pi n^2} \quad \textbf{(2)}$$

Below, I plot the function *f(x)* and its Fourier series to make sure that I have computed the Fourier coefficients correctly. Notice that the Fourier series appears to be converging uniformly to *f(x).*

> $plot([f(x), Ff(x, 1), Ff(x, 2), Ff(x, 3)], x = -3\cdot\mathrm{Pi}..3\cdot\mathrm{Pi}, colour = [black, red, purple, blue]);$




From the plot above, I notice that the error is largest at the points where the derivative is discontinuous, so I can compute the error at just one of these points for increasing values of *nmax*. I choose the point *x=0,* as that is the simplest one. Below I define the error function at the origin and plot it for increasing values of *nmax*. Since the error function is only defined for integer values of *nmax*, I need to make a listplot.

> $Ef := nmax \rightarrow evalf(\mathrm{abs}(Ff(0, nmax) - f(0)));$

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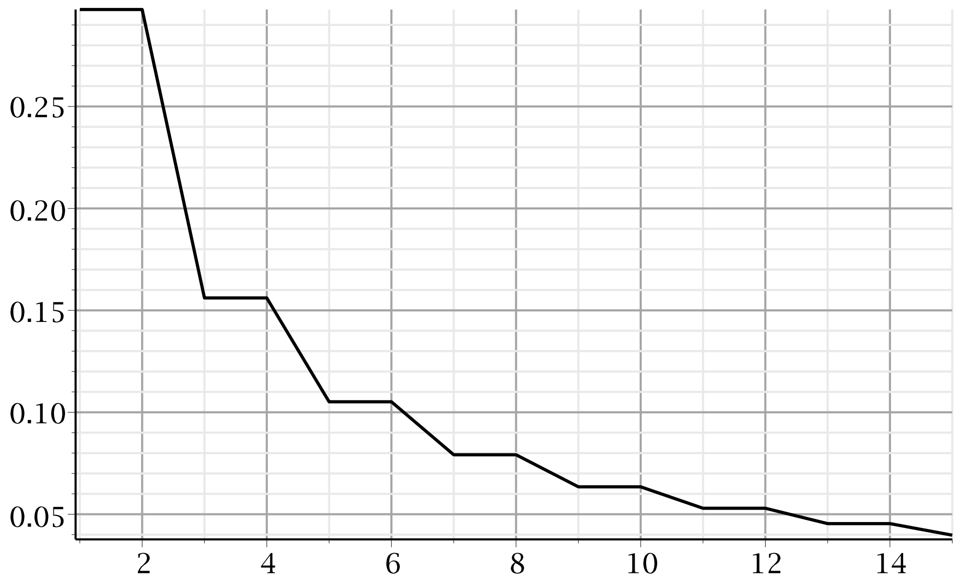
$$Ef := nmax \rightarrow evalf(|Ff(0, nmax) - f(0)|) \quad \textbf{(3)}$$

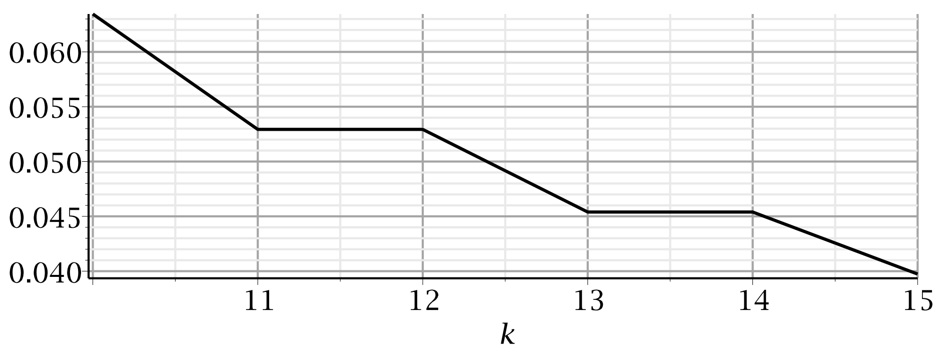
> $with(plots):$

> $listplot([seq([nmax, Ef(nmax)], nmax = 1..15)], gridlines = true);$




From the plot above, it looks as though the critical value of *nmax* is 13. Below I plot the error for *nmax*=10 to 15, so that we can see the crossing point of 0.05 more accurately.

> $\text{listplot}([\text{seq}([\text{nmax}, \text{Ef}(\text{nmax})], nmax = 10..15)], gridlines = true);$

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Now we can see clearly that if *nmax* is greater than or equal to 13, the error is below 0.05, as required.