A#3

1. Solve the IBVP

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{5}\dfrac{\partial^2 u}{\partial x^2} & \text{on } 0 < x < 2\pi,\ t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(2\pi,t) = 0 & \text{for } t > 0 \\ u(x,0) = -|x-\pi| + \pi = f(x) & \text{for } 0 < x < 2\pi \end{cases}$$

Sol'n

Step 1: Separate – assume $u(x,t) = X(x)T(t)$

$$XT' = \frac{1}{5}X''T \iff \frac{X''}{X} = \frac{5T'}{T} = -\lambda$$

Step 2: ODEs

(A) $X'' + \lambda X = 0, \quad X'(0) = X'(2\pi) = 0$

(B) $T' + \dfrac{\lambda}{5}T = 0$

Step 3: Solve ODEs

(A) i) If $\lambda < 0$, let $\lambda = -\omega^2$, then $X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$

$$X'(x) = \omega c_1 e^{\omega x} - \omega c_2 e^{-\omega x}$$

Apply BCs: $\begin{cases} X'(0) = 0 \\ X'(2\pi) = 0 \end{cases} \iff \begin{cases} c_1 - c_2 = 0 \\ \omega c_1 \left(e^{2\pi\omega} - e^{-2\pi\omega}\right) = 0 \end{cases}$

$$\iff \begin{cases} c_1 = c_2 \\ \omega = 0 \text{ or } c_1 = 0 \end{cases}$$

$\therefore$ only trivial solutions are possible in this case.

ii) If $\lambda = 0$, then $X(x) = c_1 + c_2 x$

$$X'(x) = c_2$$

Apply BCs: $c_2 = 0$ $\therefore$ $X(x) = c_1$. In this case, $u(x,t) = c_1 T(t)$ and the PDE becomes $c_1 T' = 0$, $\Leftrightarrow T(t) = c_3$ and $u(x,t) = c_1 \cdot c_3$; a constant.

$$\omega_0 = 0, \; X_0 = c_0$$

iii) If $\lambda > 0$, let $\lambda = \omega^2$, then

$$X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$$

$$X'(x) = -\omega c_1 \sin(\omega x) + \omega c_2 \cos(\omega x)$$

Apply BCs:

$$\begin{cases} X'(0) = 0 \\ X'(2\pi) = 0 \end{cases} \Leftrightarrow \begin{cases} \omega c_2 = 0 \\ -\omega c_1 \sin(2\pi\omega) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_2 = 0 \\ 2\pi\omega = n\pi, \quad n \in \mathbb{N} \end{cases}$$

$$\therefore \; \omega_n = \frac{n}{2} \;\&\; X_n = c_n \cos\left(\frac{nx}{2}\right)$$

(3) $T_n' + \frac{n^2}{20} T_n = 0 \Leftrightarrow T_n = d_n e^{-\frac{n^2}{20}t}$

Step 4: Superposition

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=0}^{\infty} c_n \cos\left(\frac{nx}{2}\right) d_n e^{-\frac{n^2}{20}t}$$

$$= \sum_{n=0}^{\infty} a_n \cos\left(\frac{nx}{2}\right) e^{-\frac{n^2}{20}t}$$

where $a_n = c_n \cdot d_n$.

Step 5: Apply ICs

$$u(x,0) = -|x-\pi| + \pi = \sum_{n=0}^{\infty} a_n \cos\left(\frac{nx}{2}\right)$$

The coefficients a_n are thus the coefficients of the Fourier cosine series for the initial condition function.

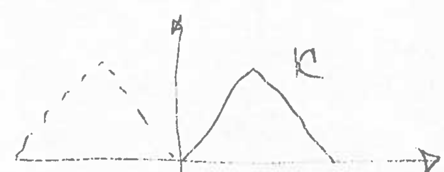
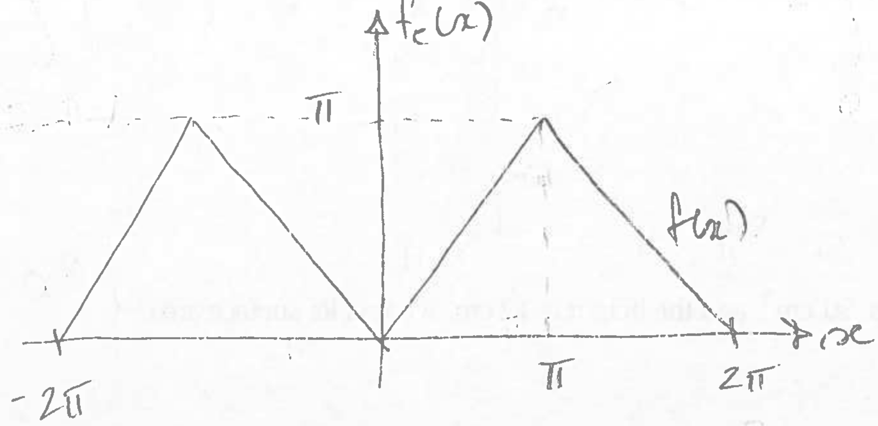


image for Fourier cosine series

We extend $f(x)$ as an even function:



$$a_n = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \underbrace{\underbrace{f_c(x)}_{\text{even}} \underbrace{\cos\left(\frac{nx}{2}\right)}_{\text{even}}}_{\text{even}} dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{nx}{2}\right)dx$$

$n \neq 0$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos\left(\frac{nx}{2}\right)dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(-x+2\pi)\cos\left(\frac{nx}{2}\right)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos\left(\frac{nx}{2}\right) - \frac{1}{\pi}\int_{\pi}^{2\pi} x\cos\left(\frac{nx}{2}\right) + 2\int_{\pi}^{2\pi}\cos\left(\frac{nx}{2}\right)dx$$

$$= \frac{1}{\pi}\left[\frac{4}{n^2}\cos\left(\frac{nx}{2}\right) + \frac{2x}{n}\sin\left(\frac{nx}{2}\right)\right]_0^{\pi}$$

$$- \frac{1}{\pi}\left[\frac{4}{n^2}\cos\left(\frac{nx}{2}\right) + \frac{2x}{n}\sin\left(\frac{nx}{2}\right)\right]_{\pi}^{2\pi} + \frac{4}{n}\sin\left(\frac{nx}{2}\right)\Bigg|_{\pi}^{2\pi}$$

$$\therefore a_n = \frac{4}{n^2\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2\pi}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi}$$

$$- \frac{4}{n^2\pi}\cos\left(\frac{2\pi n}{2}\right) - \cancel{\frac{2(2\pi)}{n\pi}\sin\left(\frac{2n\pi}{2}\right)} \to 0$$

$$+ \frac{4}{n^2\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2\pi}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$+ \cancel{\frac{4}{n}\sin\left(\frac{2n\pi}{2}\right)} \to 0 - \frac{4}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$= \cos\left(\frac{n\pi}{2}\right)\left[\frac{4}{n^2\pi} + \frac{4}{n^2\pi}\right] + \cos(n\pi)\left[-\frac{4}{n^2\pi}\right] - \frac{4}{n^2\pi}$$

$$+ \cancel{\sin\left(\frac{n\pi}{2}\right)\left[\frac{2}{n} + \frac{2}{n} - \frac{4}{n}\right]} \to 0$$

$$= \frac{4}{n^2\pi}\left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right] - \frac{4}{n^2\pi}$$

$$= \frac{4}{n^2\pi}\left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right]$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-2\pi}^{2\pi} \underbrace{f_e(x)}_{\text{even}}\,dx\right] = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} x\,dx + \frac{2}{2\pi}\int_{\pi}^{2\pi}(-x+2\pi)\,d$$

$$= \frac{2}{2\pi}\,\frac{x^2}{2}\Big|_0^{\pi} + \frac{2}{2\pi}\left(-\frac{x^2}{2} + 2\pi x\right)\Big|_{\pi}^{2\pi}$$

$$= \frac{x^2}{2\pi}\Big|_0^{\pi} + \left(-\frac{x^2}{2\pi} + 2x\right)\Big|_{\pi}^{2\pi}$$

$$= \left(\frac{\pi^2}{2\pi} - 0\right) + \left(-\frac{4\pi^2}{2\pi} + \frac{8\pi}{2} + \frac{\pi^2}{2\pi} - \frac{4\pi}{2}\right)$$

$$= \frac{\pi}{2} - \cancel{\frac{4\pi}{2}} + \cancel{\frac{8\pi}{2}} + \frac{\pi}{2} - \cancel{\frac{4\pi}{2}}$$

$$= \pi$$

$$\therefore\ f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right]$$

Step 6: Final Answer

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right]\cos\left(\frac{nx}{2}\right)e^{-\frac{n^2}{32}t}$$

2. $$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0 & \\ u(\pi,t) + \dfrac{\partial u}{\partial x}(\pi,t) = 0 & t > 0 \\ u(x,0) = g(x) & 0 < x < \pi \end{cases}$$

Step 1: Separate

Let $u(x,t) = X(x)T(t)$. Then the PDE becomes

$$XT' = X''T \iff \underbrace{\frac{T'}{T}}_{\text{only } t\text{-dependence}} = \underbrace{\frac{X''}{X}}_{\text{only } x\text{-dependence}} = -\lambda$$

∵ x + t are independent variables, the ratios T'/T + X''/X can only be equal if they each equal a constant, which we call $-\lambda$.

Step 2: ODEs

(A) $$\begin{cases} X'' + \lambda X = 0, \\ X(0) = 0, \quad X(\pi) + X'(\pi) = 0 \end{cases}$$

(B) $T' + \lambda T = 0$

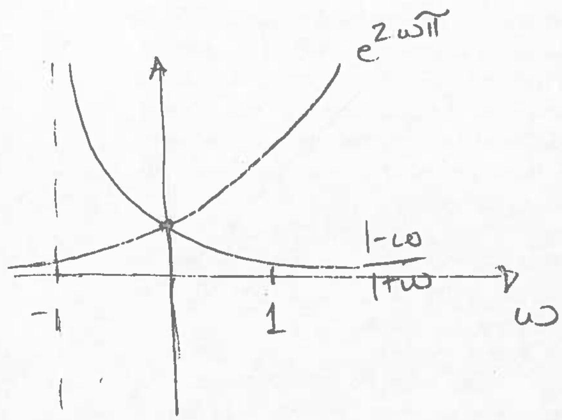
Step 3: solve (A) + (B)

(A) Case 1: $\lambda = -\omega^2 < 0$

Then $X(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$

Apply BCs.

$$\begin{cases} X(0) = 0 \\ X(\pi) + X'(\pi) = 0 \end{cases} \iff \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\omega\pi} + c_2 e^{-\omega\pi} + \omega c_1 e^{\omega\pi} - \omega c_2 e^{-\omega\pi} = 0 \end{cases}$$



$$\iff \begin{cases} c_1 = -c_2 \\ c_2\left[-(1+\omega)e^{\omega\pi} + (1-\omega)e^{-\omega\pi}\right] = 0 \end{cases}$$

$$\iff \begin{cases} c_1 = -c_2 \\ c_2 = 0 \text{ or } e^{2\omega\pi} = \frac{(1-\omega)}{1+\omega} \end{cases}$$

We see that there is no intersection btw the functions $y = e^{2\omega\pi}$ & $y = \frac{1-\omega}{1+\omega}$ except at $\omega = 0$, & so we have only trivial solutions.

Case 2: $\lambda = 0$

Then $X(x) = c_1 x + c_2$.

Apply BCs:

$$\begin{cases} X(0) = 0 \\ X(\pi) + X'(\pi) = 0 \end{cases} \iff \begin{cases} c_2 = 0 \\ c_1\pi + c_1 = 0 \end{cases} \iff \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases}$$

$\therefore$ we only have trivial solutions in this case.

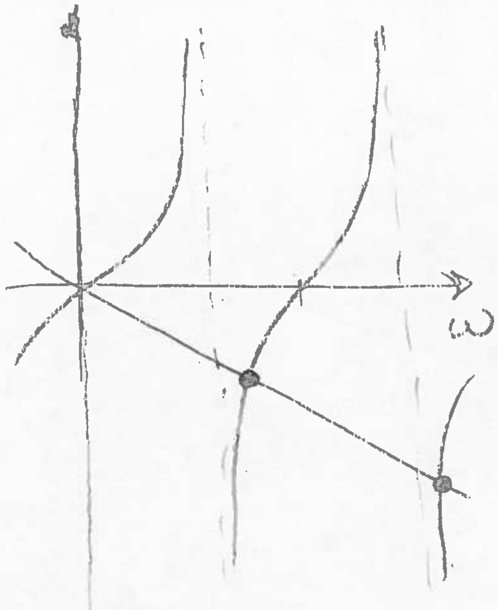
Case 3: $\lambda = \omega^2 > 0$

Then $X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$.

Apply BCs:

$$\begin{cases} X(0)=0 \\ X(\pi)+X'(\pi)=0 \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 \sin(\omega\pi) + \omega c_2 \cos(\omega\pi) = 0 \end{cases}$$

$$\iff \begin{cases} c_1 = 0 \\ c_2 = 0 \text{ or } \tan(\omega\pi) = -\omega \end{cases}$$



We see that there are infinitely many values $\omega_n > 0$ for which the functions $y = \tan(\pi\omega)$ and $y = -\omega$ intersect.

$\therefore$ the eigenvalues ω_n are the solutions of $\tan(\pi\omega_n) = -\omega_n$, and the eigenfunctions are

$$X_n(x) = c_n \sin(\omega_n x).$$

(B) $T' + \lambda T = 0 \Rightarrow T_n' + \omega_n^2 T_n = 0 \iff T_n(t) = d_n e^{-\omega_n^2 t}$

Step 4: Superposition

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\omega_n x)\, d_n e^{-\omega_n^2 t}$$

Let $b_n = c_n d_n$, then

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n x) e^{-\omega_n^2 t}$$

where ω_n is the solutions of

$$\tan(\pi \omega_n) = -\omega_n .$$

Step 5: Apply the IC

$$u(x,0) = g(x) \Leftrightarrow \sum_{n=1}^{\infty} b_n \sin(\omega_n x) = g(x)$$

The coefficients b_n are given by the Fourier sine series for $g(x)$. We thus extend $g(x)$ as an odd function $g_o(x)$ & obtain

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\underbrace{g_o(x)}_{\text{odd}} \underbrace{\sin(\omega_n x)}_{\text{odd}}}_{\text{even}} dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} g(x) \sin(\omega_n x) \, dx$$

$\therefore$ the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n x)\, e^{-\omega_n^2 t}$$

where the eigenvalues ω_n are the solutions of

$$\tan(\omega_n \pi) = -\omega_n$$

and the coefficients b_n are given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(\omega_n x)\,dx .$$

3.
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + P(x) & 0 < x < L, \quad t > 0 \quad \ldots \text{(1a)} \\ u(0,t) = K, \quad \dfrac{\partial u}{\partial x}(L,t) = u(L,t) & t > 0 \quad \ldots \text{(1b)} \\ u(x,0) = h(x) & 0 < x < L \quad \ldots \text{(1c)} \end{cases}$$

a) Physical interpretation:

- At $x=0$, the bar is held to the constant temperature K.
- At $x=L$, the bar is partially insulated. Insulation decreases as temperature increases.
- At $t=0$ (the starting time), the temperature distribution in the rod is given by $h(x)$.

b) Assume

$$u(x,t) = \underbrace{w(x,t)}_{\text{transient}} + \underbrace{v(x)}_{\text{steady-state}} \quad \ldots \text{(2)}$$

Then, (1) becomes

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} + P(x) \quad \ldots \ldots \text{(3a)}$$

$$w(0,t) + v(0) = K \quad , \quad \frac{\partial w}{\partial x}(L,t) + \frac{\partial v}{\partial x}(L) = w(L,t) + v(L) \quad \ldots \quad (3b)$$

$$w(x,0) + v(x) = h(x) \quad \ldots \quad (3c)$$

If we let $t \to \infty$, then (3a) & (3b) become

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} + P(x) = 0 \\ v(0) = K \quad , \quad v'(L) = v(L) \end{cases} \quad \ldots \quad (4)$$

Solving (4a) we obtain

$$v''(x) = -P(x) \Leftrightarrow v(x) = -\int\left(\int P(x)\,dx\right)dx + Cx + D \quad \ldots \quad (5)$$

where the constants of integration are determined by 4(b).

Plugging (4) into (3) we obtain a homogeneous PDE problem for $w(x,t)$:

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} & \ldots \quad (6a) \\ w(0,t) = 0, \quad \dfrac{\partial w}{\partial x}(L,t) = w(L,t) & \ldots \quad (6b) \\ w(x,0) = h(x) - v(x) & \ldots \quad (6c) \end{cases}$$

step 1: Separate

$w(x,t) = X(x)\,T(t)$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \Leftrightarrow XT' = X''T \Leftrightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$$

step 2: ODEs

Ⓐ $\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0,\ X'(L) = X(L) \end{cases}$ Ⓑ $T' + \lambda T = 0$

step 3: solve Ⓐ + Ⓑ

Ⓐ case 1 $\lambda < 0$ $\lambda = -\eta^2$

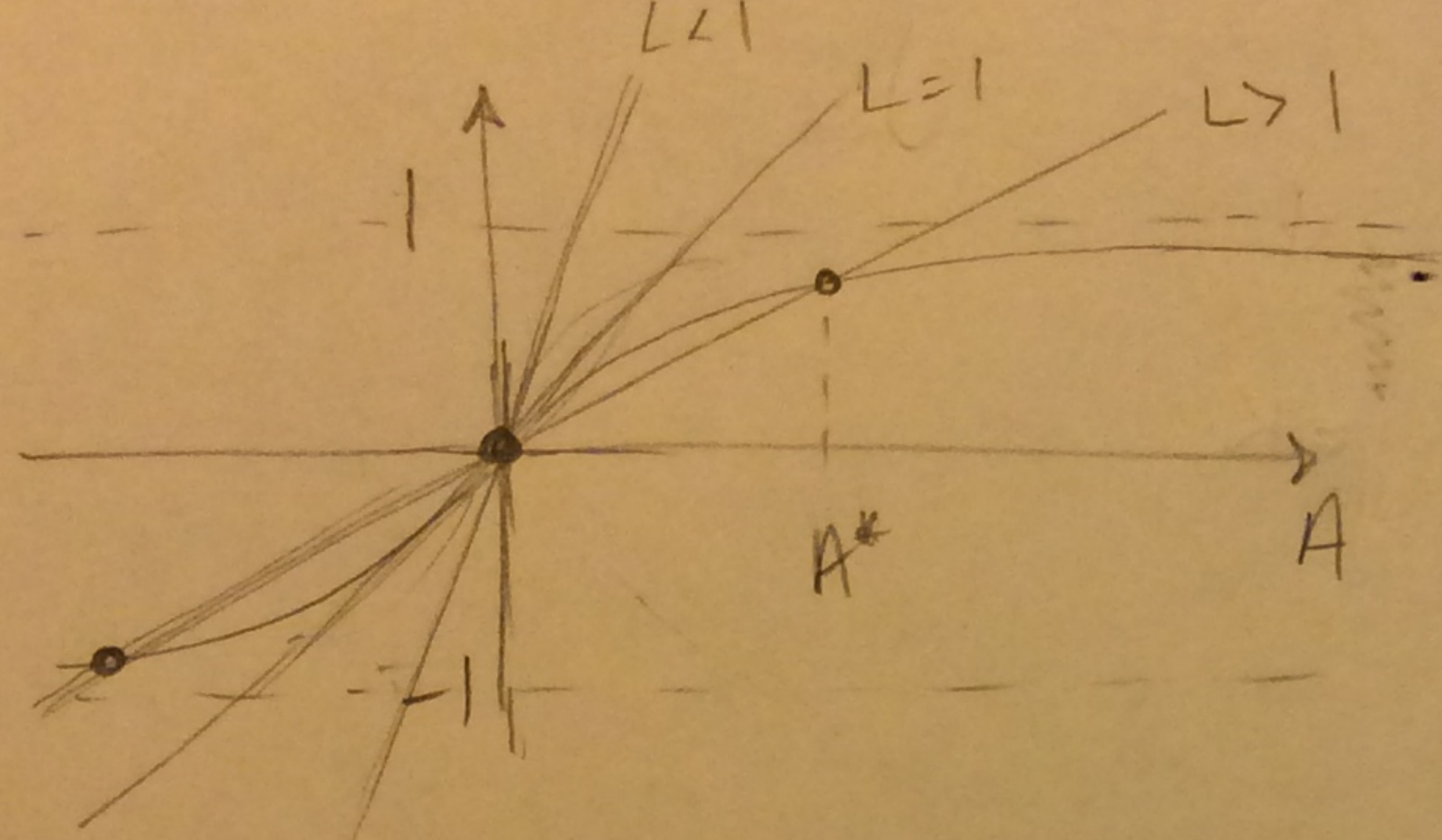
Then $X(x) = C_1 e^{\eta x} + C_2 e^{-\eta x}$

BCs: $\begin{cases} X(0) = 0 \\ X'(L) = X(L) \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = 0 \\ \eta C_1 e^{\eta L} - \eta C_2 e^{-\eta L} = C_1 e^{\eta L} + C_2 e^{-\eta L} \end{cases} \Leftrightarrow //$

$// \Leftrightarrow \begin{cases} C_1 = -C_2 \\ -\eta C_2 e^{\eta L} - \eta C_2 e^{-\eta L} = -C_2 e^{\eta L} + C_2 e^{-\eta L} \end{cases}$

$\Leftrightarrow \begin{cases} C_1 = -C_2 \\ C_2 = 0 \text{ or } -\eta\cosh(\eta L) = -\sinh(\eta L) \end{cases}$

$$\Leftrightarrow \begin{cases} C_1 = -C_2 \\ C_2 = 0 \text{ or } \tanh(\eta L) = \eta \end{cases} \Leftrightarrow \tanh(A) = \frac{A}{L}$$



i) If $L \leq 1$, we only have trivial solutions.

ii) If $L > 1$, there is a nontrivial solution, as shown in the sketch above. Let $\eta^* = \frac{A^*}{L}$. Then the solution is

$$X^*(x) = C_1 e^{\eta^* x} - C_1 e^{-\eta^* x} = \tilde{C}_1 \sinh(\eta^* x)$$

Case 2: $\lambda = 0$

Then $X(x) = C_1 x + C_2$

$$\text{BCs: } \begin{cases} X(0) = 0 \\ X'(L) = X(L) \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ C_1 = C_1 L \end{cases} \Leftrightarrow \begin{cases} C_2 = 0 \\ C_1 = 0 \end{cases}$$

We only have trivial solutions in this case.

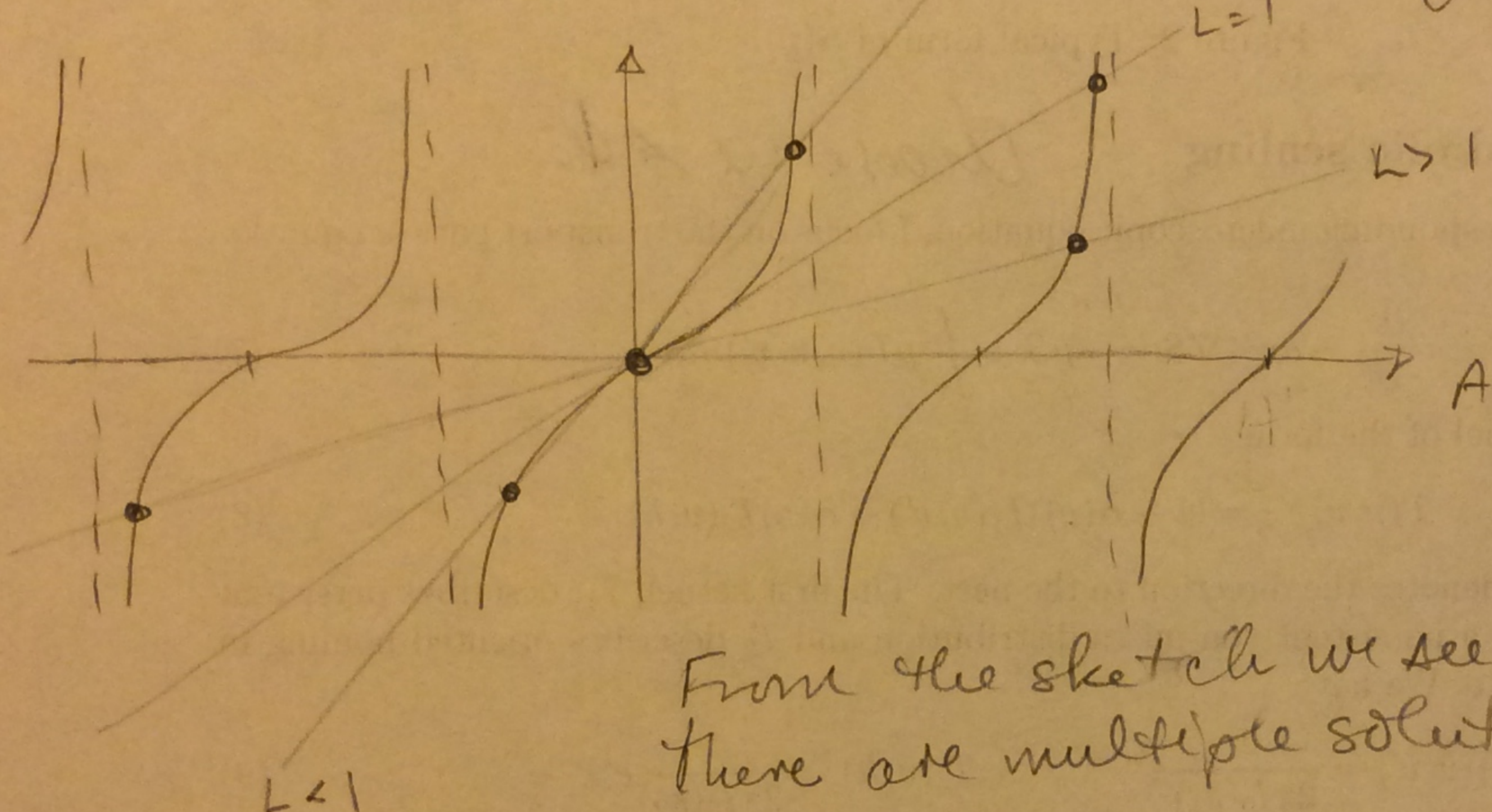
Case 3: $\lambda > 0$, $\lambda = \eta^2$

Then $X(x) = c_1 \cos(\eta x) + c_2 \sin(\eta x)$

$$\text{BCs: } \begin{cases} X(0) = 0 \\ X'(L) = X(L) \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 \eta \cos(\eta L) = c_2 \sin(\eta L) \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \text{ OR } \eta = \tan(\eta L) \Leftrightarrow \frac{A}{L} = \tan(A) \end{cases}$$

.....(7)



From the sketch we see that there are multiple solutions

$$A_n = L\eta_n \Leftrightarrow \eta_n = \frac{A_n}{L}$$

for all values of $L > 0$. The solution for each n is

$$X_n(x) = \tilde{c}_n \sin(\eta_n x).$$

B case 1 $\lambda = -\eta^{*2}$

$$T' - \eta^{*2} T = 0 \iff T = T_0 e^{\eta^{*2} t}$$

case 2 $\lambda = \eta_n^2$

$$T' + \eta_n^2 T = 0 \iff T = T_0 e^{-\eta_n^2 t}$$

Step 4: Superposition

Case 1: $\lambda = -\eta^{*2}$

$$w(x,t) = C \sinh(\eta^* x) e^{\eta_*^2 t}$$

Case 2: $\lambda = \eta_n^2$

$$w(x,t) = \sum_{n=1}^{\infty} \left(c_n \sin(\eta_n x) e^{-\eta_n^2 t} \right)$$

Step 5: Apply ICs

$u(x,0) = h(x) - v(x)$

Case 1: $\lambda = -\eta^{*2}$

$$w(x,0) = C\sinh(\eta^* x) = h(x) - v(x)$$

In general, it will not be possible to find a constant C such that this equation is satisfied, & so we have no solution in this case.

Case 2: $\lambda = \eta_n^2$

$$w(x,0) = \sum_{n=1}^{\infty} c_n \sin(\eta_n x) = h(x) - v(x)$$

The coefficients c_n are given by

$$c_n = \frac{2}{L}\int_0^L (h(x) - v(x))\,dx, \qquad \ldots\ (8)$$

and so we do have a solution in this case.

The formal solution for $u(x,t)$ is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\eta_n x) e^{-\eta_n^2 t} + v(x)$$

where c_n is given by (8) & $v(x)$ is given by (5).

Lindsey Reinholz

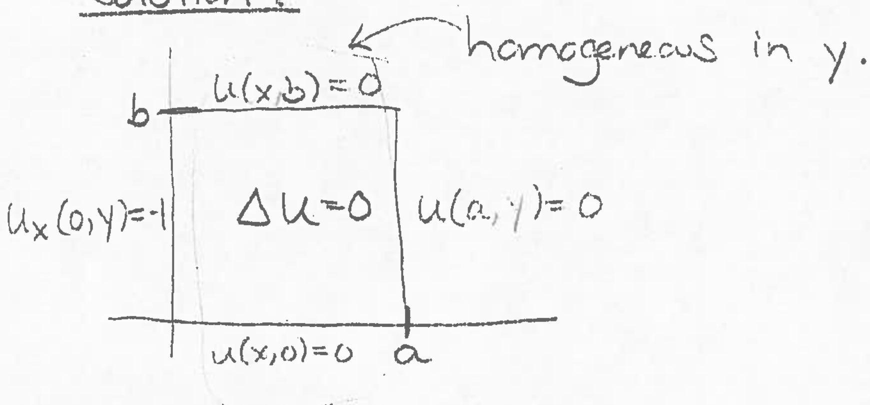
4. Find a formal solution to the given boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } 0<x<a \cap 0<y<b$$

$u(x,0) = u(x,b) = 0$ in $0<x<a$.

$u(a,y) = 0$ and $u_x(0,y) = -1$ in $0<y<b$.

Solution:



Set $u(x,y) = F(x)G(y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$F''(x)G(y) + F(x)G''(y) = 0.$$

$$\frac{F''(x)}{F(x)} = \frac{-G''(y)}{G(y)} = \lambda.$$

*Note that we have chosen a positive constant to give non-negative eigenvalues for $G(y)$ because the boundary conditions in y are homogeneous.

① $G''(y) + \lambda G(y) = 0$
$G(0) = G(b) = 0$

② $F''(x) - \lambda F(x) = 0.$
$F(a) = 0$

Look at ①:

Set $G = e^{ry}$

$G' = re^{ry}$

$G'' = r^2 e^{ry}$

$$r^2 e^{ry} + \lambda e^{ry} = 0$$
$$r^2 = -\lambda$$
$$r = \pm\sqrt{-\lambda}$$

Case 1: $\lambda = 0$

$G''(y) = 0$

$G'(y) = Ay$

$G(y) = Ay + B$

$G(0) = B = 0 \quad \therefore G(y) = Ay$

$G(b) = Ab = 0 \Rightarrow A = 0 \quad \therefore G(y) = 0$

Trivial Solution.

Case 2: $\lambda < 0$ set $\lambda = -\mu^2$ with $\mu > 0 \to r = \pm\mu$.

$G(y) = c_1 \cosh \mu y + c_2 \sinh \mu y$

$G(0) = c_1(1) + c_2(0) = 0$

$\Rightarrow c_1 = 0$

$\therefore G(y) = c_2 \sinh \mu y$

$G(b) = c_2 \underbrace{\sinh \mu b}_{\neq 0 \text{ since } \mu > 0} = 0$

$\therefore c_2 = 0$ Trivial Solution.

Case 3: $\lambda > 0$ set $\lambda = \mu^2$ with $\mu > 0 \to r = \pm i\mu$.

$G(y) = c_3 \cos \mu y + c_4 \sin \mu y$

$G(0) = c_3(1) + c_4(0) = 0$

$\Rightarrow c_3 = 0$

$\therefore G(y) = c_4 \sin \mu b = 0$

$c_4 = 0$ Trivial Solution

$\sin \mu b = 0$

$\mu b = n\pi \quad n = 1, 2, \ldots$

$\mu = \frac{n\pi}{b}$

2013

(6)

Oct 21st

Lindsey Reinholz

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{b}\right)^2 \quad n = 1, 2, \ldots$

Eigenfunctions: $G_n(y) = c_n \sin \frac{n\pi}{b} y$

Look at (2): $F''(x) - \lambda F(x) = 0 \qquad F(a) = 0$

Set $F = e^{sx}$

$F' = se^{sx}$

$F'' = s^2 e^{sx} \qquad s^2 e^{sx} - \lambda e^{sx} = 0$

$s^2 = \lambda$

$s = \pm\sqrt{\lambda} = \pm \frac{\pi n}{b}$ ← real so get hyperbolics

$$F(x) = c_5 \cosh\left(\frac{n\pi}{b} x\right) + c_6 \sinh\left(\frac{n\pi}{b}(a-x)\right)$$

$$F(a) = c_5 \cosh\left(\frac{n\pi}{b} a\right) + 0 = 0$$

$$\Rightarrow c_5 = 0$$

$$\therefore F_n(x) = d_n \sinh\left(\frac{n\pi}{b}(a-x)\right)$$

Superposition gives

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{b} y \; d_n \sinh\left(\frac{n\pi}{b}(a-x)\right)$$

$$= \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \sinh\left(\frac{n\pi}{b}(a-x)\right) \qquad \text{where } a_n = c_n d_n$$

$$u_x(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \left(\frac{-n\pi}{b}\right) \cosh\left(\frac{n\pi}{b}(a-x)\right)$$

$$u_x(0,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \left(\frac{-n\pi}{b}\right) \cosh\left(\frac{n\pi a}{b}\right) = -1$$

$$1 = \sum_{n=1}^{\infty} \left(a_n \frac{n\pi}{b} \cosh\left(\frac{n\pi a}{b}\right)\right) \sin \frac{n\pi y}{b}.$$

$$a_n \frac{n\pi}{b} \cosh \frac{n\pi a}{b} = \frac{2}{b}\int_0^b \sin \frac{n\pi y}{b}\, dy$$

$$= \frac{-2}{\cancel{b}} \frac{\cancel{b}}{n\pi} \cos \frac{n\pi y}{b}\Big|_0^b$$

$$= \frac{-2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}(1-\cos n\pi)$$

$$= \frac{2}{n\pi}(1-(-1)^n)$$

$$\therefore a_n = \frac{2/n\pi\,(1-(-1)^n)}{n\pi/b \cosh(n\pi a/b)} = \frac{2b(1-(-1)^n)}{(n\pi)^2\cosh(n\pi a/b)}$$

When $n=2k$ for $k=1,2,\ldots$, then $(1-(-1)^n)=0$, so $a_n=0$

$n=2k-1$ for $k=1,2,\ldots$, then $(1-(-1)^n)=2$, so

$$a_n = \frac{4b}{(2k-1)^2\pi^2\cosh((2k-1)\pi a/b)}$$

so

$$u(x,y)=\sum_{k=1}^{\infty} \frac{4b}{(2k-1)^2\pi^2\cosh((2k-1)\pi a/b)} \sinh\left(\frac{(2k-1)}{b}\pi(a-x)\right)\sin\left(\frac{(2k-1)\pi y}{b}\right)$$