Math 319 - Differential Equations II
Assignment # 4

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

1. Consider the following non-homogeneous wave equation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 7e^t, \qquad 0 < x < 1, \quad t > 0, \tag{1a}$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{1b}$$

$$u(x,0) = 5\sin(3\pi x), \qquad 0 < x < 1 \tag{1c}$$

$$\frac{\partial u}{\partial t}(x,0) = 8\sin(2\pi x), \qquad 0 < x < 1 \tag{1d}$$

$$\tag{1e}$$

We would like to find a solution to (1) of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x), \tag{2}$$

where the $u_n(t)$ are functions to be determined.

(a) For each fixed t, write $h(x,t) = 7e^t$ as a Fourier sine series and compute the coefficients for this series.

(b) Substitute the Fourier series for $u(x,t)$ and $h(x,t)$ back into the original PDE and derive a non-homogeneous, constant coefficient ODE for $u_n(t)$.

(c) Solve the ODE for $u_n(t)$ using variation of parameters.

(d) Apply the initial conditions to find the coefficients of the Fourier series in $u(x,t)$, and give the full solution.

2. Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{3a}$$

$$u(x,0) = \begin{cases} 2(1-x^2), & \text{if } -1 < x < 1, \\ 0, & \text{else }. \end{cases} \qquad -\infty < x < \infty \tag{3b}$$

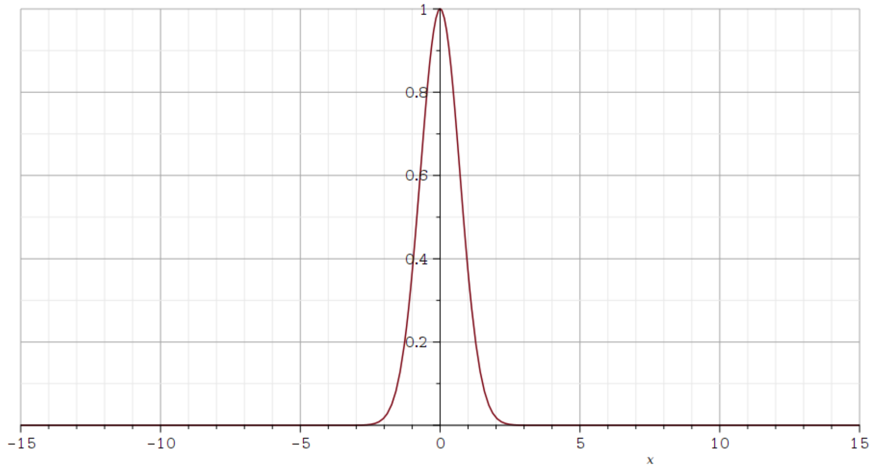
$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad -\infty < x < \infty. \tag{3c}$$

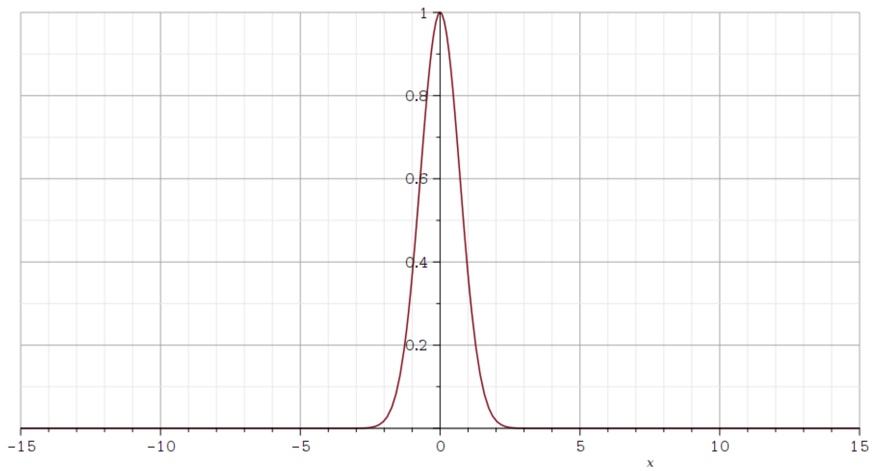
$$\tag{3d}$$

(a) Find the solution.

(b) Plot the solution at $t = 0$, 2, and 4 (by hand or using Maple - your choice!).

3. Consider an infinite domain where the quantity of interest, $u(x,t)$ satisfies $u(x,0) = f(x)$, shown in the plots below (i.e., both plots below are plots of $f(x)$).





On the upper plot, assume that $u(x,t)$ is governed by the heat equation, and on the lower plot, assume that $u(x,t)$ is governed by the wave equation. On each plot, sketch the solution at two future timepoints, showing how the initial conditions evolve under the two different PDEs. Then describe the physical scenario that is modeled in each case.