Math 319 - Differential Equations II
Assignment # 4
due Mon Nov 14th, 11:30am, SCI 386

Instructions: You are being evaluated on the presentation, as well as the correctness, of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark.

The assignment may be done with up to 4 other classmates (i.e. total group size: no more than 5). If you collaborate with classmates, the group should hand in one document with all contributing names at the top.

1. consider teh following non-homogeneous wave equation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 7e^t, \qquad 0 < x < 1, \quad t > 0, \tag{1a}$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{1b}$$

$$u(x,0) = 5\sin(3\pi x), \qquad 0 < x < 1 \tag{1c}$$

$$\frac{\partial u}{\partial t}(x,0) = 8\sin(2\pi x), \qquad 0 < x < 1 \tag{1d}$$

$$\tag{1e}$$

   We would like to find a solution to (1) of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x), \tag{2}$$

   where the $u_n(t)$ are functions to be determined.

   (a) For each fixed t, write $h(x,t) = 7e^t$ as a fourier sine series and compute the coefficients for this series.

   (b) Substitute the Fourier series for $u(x,t)$ and $h(x,t)$ back into the original PDE and derive a non-homogeneous, constant coefficient ODE for $u_n(t)$.

   (c) Solve the ODE for $u_n(t)$ using variation of parameters.

   (d) Apply the initial conditions to find the coefficients of the Fourier series in $u(x,t)$, and give the full solution.

2. Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0, \tag{3a}$$

$$u(x,0) = \begin{cases} 2(1-x^2), & \text{if } -1 < x < 1, \\ 0, & \text{else .} \end{cases} \qquad -\infty < x < \infty \tag{3b}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad -\infty < x < \infty. \tag{3c}$$

(a) Find the solution.

(b) Plot the solution at $t = 0$, 2, and 4 (by hand or using Maple - your choice!).

A#4 - Solutions

1. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + 7e^t & 0<x<1,\ t>0 \quad \text{(1a)} \\ u(0,t) = u(1,t) = 0 & t>0 \quad \text{(1b)} \\ u(x,0) = 5\sin(3\pi x) & 0<x<1 \quad \text{(1c)} \\ \dfrac{\partial u}{\partial t}(x,0) = 8\sin(2\pi x) & 0<x<1 \quad \text{(1d)} \end{cases}$$

Solution:

a) For each fixed t, we can compute a Fourier sine series for $h(x,t) = 7e^t$:

$$7e^t = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

with

$$h_n(t) = \frac{2}{1}\int_0^1 7e^t \sin(n\pi x)\,dx \qquad \text{for } n=1,2,3,\ldots$$

$$= 14e^t \int_0^1 \sin(n\pi x)\,dx$$

$$= 14e^t \left[\frac{-\cos(n\pi x)}{n\pi}\right]\Bigg|_0^1$$

$$= 14e^t \left[\frac{-\cos(n\pi)+1}{n\pi}\right]$$

$$= \boxed{\frac{14e^t(1-(-1)^n)}{n\pi}} \qquad \text{for } n=1,2,3,\ldots$$

b) Substitution into the PDE yields

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 7e^t$$

$$\sum_{n=1}^{\infty} u_n''(t)\sin(n\pi x) - \sum_{n=1}^{\infty} u_n(t)(n\pi)^2 \sin(n\pi x) = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

$$\sum_{n=1}^{\infty}\left[u_n''(t) + (n\pi)^2 u_n(t)\right]\sin(n\pi x) = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

Equating coefficients yields

$$\boxed{u_n''(t) + (n\pi)^2 u_n(t) = h_n(t) = \frac{14e^t(1-(-1)^n)}{n\pi}} \quad \cdots (2)$$

This is a non-homogeneous constant-coefficient equation that can be solved by using the method of variation of parameters.

1c) The corresponding homogeneous ODE is

$$u_{n_h}''(t) + (n\pi)^2 u_{n_h}(t) = 0$$

has solutions

$$u_{n_h}(t) = c_1 \sin(n\pi t) + c_2 \cos(n\pi t). \quad \cdots (3)$$

We thus try, for the particular solution,

$$u_{n_p}(t) = v_1(t)\sin(n\pi t) + v_2(t)\cos(n\pi t). \quad \cdots (4)$$

In order for (4) to satisfy (2), $v_1(t) + v_2(t)$ must satisfy

$$\begin{cases} v_1'(t)\sin(n\pi t) + v_2'(t)\cos(n\pi t) = 0 \\ v_1'(t)\, n\pi\cos(n\pi t) - v_2'(t)\, n\pi \sin(n\pi t) = h_n(t) \end{cases}$$

cos 1,

$$1/ \Leftrightarrow \begin{cases} n\pi\, v_1'(t)\,\underbrace{\left[\sin^2(n\pi t)+\cos^2(n\pi t)\right]}_{1} = h_n(t)\cos(n\pi t) \\ n\pi\, v_2'(t)\,\underbrace{\left[\cos^2(n\pi t)+\sin^2(n\pi t)\right]}_{1} = -h_n(t)\sin(n\pi t) \end{cases}$$

$$\Leftrightarrow \begin{cases} v_1'(t) = \dfrac{h_n(t)}{n\pi}\cos(n\pi t) \\[2ex] v_2'(t) = -\dfrac{h_n(t)}{n\pi}\sin(n\pi t) \end{cases}$$

$$\Leftrightarrow \begin{cases} v_1(t) = \dfrac{14(1-(-1)^n)}{(n\pi)^2}\displaystyle\int e^t\cos(n\pi t)\,dt \\[2ex] v_2(t) = \dfrac{-14(1-(-1)^n)}{(n\pi)^2}\displaystyle\int e^t\sin(n\pi t)\,dt \end{cases}$$

$$\Leftrightarrow \begin{cases} v_1(t) = \dfrac{14(1-(-1)^n)}{(n\pi)^2}\,\dfrac{1}{(n\pi)^2+1}\,e^t\left(n\pi\sin(n\pi t)+\cos(n\pi t)\right) \\[2ex] v_2(t) = \dfrac{-14(1-(-1)^n)}{(n\pi)^2}\,\dfrac{1}{(n\pi)^2+1}\,e^t\left(\sin(n\pi t)-n\pi\cos(n\pi t)\right. \end{cases}$$

$$\therefore u_{n_p}(t) = v_1(t)\sin(n\pi t) + v_2(t)\cos(n\pi t)$$

$$= \frac{14(1-(-1)^n)}{(n\pi)^2}\frac{1}{(n\pi)^2+1}e^t\left[n\pi\left(\sin^2(n\pi t)+\cos^2(n\pi t)\right)\right]$$

$$= \frac{14(1-(-1)^n)}{n\pi((n\pi)^2+1)}e^t \quad \dots (5)$$

Combining (5) + (3) we obtain

$$u_n(t) = u_{n_h}(t) + u_{n_p}(t)$$

$$= c_1\sin(n\pi t) + c_2\cos(n\pi t) + \underbrace{\frac{14(1-(-1)^n)}{n\pi[(n\pi)^2+1]}}_{A_n}e^t \quad \dots (6)$$

1d) Thus, the full solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_{1n}\sin(n\pi t) + c_{2n}\cos(n\pi t) + A_n e^t\right)\sin(n\pi x) \quad \dots (7)$$

We now apply the initial conditions in order to determine c_{1n} and c_{2n}:

1^{st} BC

$$u(x,0) = 5\sin(3\pi x) \Leftrightarrow \sum_{n=1}^{\infty}\left(C_{2_n} + A_n\right)\sin(n\pi x) = 5\sin(3\pi x)$$

By orthogonality, we conclude that

$$\begin{cases} C_{2_n} + A_n = 0 \Leftrightarrow C_{2_n} = -A_n, \quad \forall n \neq 3 \\ C_{2_3} + A_3 = 5 \Leftrightarrow C_{2_3} = -A_3 + 5 \end{cases} \quad \ldots (8)$$

2^{nd} BC

$$\frac{\partial u}{\partial t}(x,0) = 8\sin(2\pi x) \Leftrightarrow \sum_{n=1}^{\infty}\left[C_{1_n} n\pi + A_n\right]\sin(n\pi x) = 8\sin(2\pi x)$$

By orthogonality, we conclude that

$$\begin{cases} C_{1_n} n\pi + A_n = 0 \Leftrightarrow C_{1_n} = \dfrac{-A_n}{n\pi}, \quad \forall n \neq 2 \\ C_{1_2}(2\pi) + A_2 = 8 \Leftrightarrow C_{1_2} = -\dfrac{A_2}{2\pi} + \dfrac{8}{2\pi} \end{cases} \quad \ldots (9)$$

Note that in (8) & (9), the terms C_{2_3} & C_{1_2} have the same form as the terms C_{2_n} & C_{1_n}, but with an added constant. So if we separate out this added constant we can simplify $u(x,t)$.

Putting these results all together we find that

$$\boxed{\begin{aligned} u(x,t) &= \frac{4}{\pi}\sin(2\pi t)\sin(2\pi x) + 5\cos(3\pi t)\sin(3\pi x) \\ &\quad + \sum_{n=1}^{\infty} A_n\left(e^{t} - \sin(n\pi t) - \cos(n\pi t)\right)\sin(n\pi x) \\ \text{where} \\ A_n &= \frac{14\left(1-(-1)^n\right)}{n\pi\left[(n\pi)^2+1\right]}. \end{aligned}}$$

2.
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2} \qquad 0<x<1,\ t>0 & \cdots (3a) \\[2ex] u(x,0) = \begin{cases} 2(1-x^2), & \text{if } -1<x<1 \\ 0, & \text{else} \end{cases} & \cdots (3b) \\[2ex] \dfrac{\partial u}{\partial t}(x,0) = 0 & \cdots (3c) \end{cases}$$

a) The general solution is

$$u(x,t) = A(x+\alpha t) + B(x-\alpha t)$$

where $\alpha = 2$.

We need to choose the functions A & B such that the ICs are satisfied.

$u(x,0) = A(x) + B(x) = f(x)$ where $f(x) = \begin{cases} 2(1-x^2) & \text{if } -1<x<1 \\ 0 & \text{else} \end{cases}$

$$u_t(x,t) = 2A'(x+2t) - 2B'(x-2t)$$

$$u_t(x,0) = 2A'(x) - 2B'(x) = 0$$

$$2A'(x) = 2B'(x)$$

$$A'(x) = B'(x)$$

$$A(x) = B(x) + C \qquad C = \text{constant}$$

$$A(x) - B(x) = C$$

This gives us the following system of equations

① $A(x) + B(x) = f(x)$

② $A(x) - B(x) = C$

Adding ① to ② yields

$$2A(x) = f(x) + C$$

$$A(x) = \frac{f(x)}{2} + \frac{C}{2}$$

Subtracting ② from ① yields

$$2B(x) = f(x) - C$$

$$B(x) = \frac{f(x)}{2} - \frac{C}{2}$$

Then

$$u(x,t) = A(x+2t) + B(x-2t)$$

$$= \frac{f(x+2t)}{2} + \frac{C}{2} + \frac{f(x-2t)}{2} - \frac{C}{2}$$

$$= \frac{1}{2}\left[f(x+2t) + f(x-2t)\right]$$

Where

$$f(x+2t) = \begin{cases} 2(1-(x+2t)^2) & \text{if } -1<x+2t<1 \\ 0 & \text{else} \end{cases}$$

and

$$f(x-2t) = \begin{cases} 2(1-(x-2t)^2) & \text{if } -1<x-2t<1 \\ 0 & \text{else} \end{cases}$$

b) At $t=0$

$$u(x,0)=\frac{1}{2}[f(x)+f(x)]$$
$$=f(x)$$
$$=\begin{cases}2(1-x^2) & -1<x<1\\ 0 & \text{else}\end{cases}$$

At $t=2$:

$$u(x,2)=\frac{1}{2}[f(x+4)+f(x-4)]$$

where

$$f(x-4)=\begin{cases}2(1-(x-4)^2) & \text{if } -1<x-4<1\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}2(1-(x^2-8x+16)) & \text{if } 3<x<5\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}-2(x^2-8x+15) & \text{if } 3<x<5\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}-2(x-3)(x-5) & \text{if } 3<x<5\\ 0 & \text{else}\end{cases}$$

and

$$f(x+4)=\begin{cases}2(1-(x+4)^2) & \text{if } -1<x+4<1\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}2(1-(x^2+8x+16)) & \text{if } -5<x<-3\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}-2(x+3)(x+5) & \text{if } -5<x<-3\\ 0 & \text{else}\end{cases}$$

At $t=4$:

$$u(x,4)=\frac{1}{2}[f(x+8)+f(x-8)]$$

where

$$f(x-8)=\begin{cases}2(1-(x-8)^2) & \text{if } -1<x-8<1\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}2(1-(x^2-16x+64)) & \text{if } 7<x<9\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}-2(x^2-16x+63) & \text{if } 7<x<9\\ 0 & \text{else}\end{cases}$$
$$=\begin{cases}-2(x-7)(x-9) & \text{if } 7<x<9\\ 0 & \text{else}\end{cases}$$

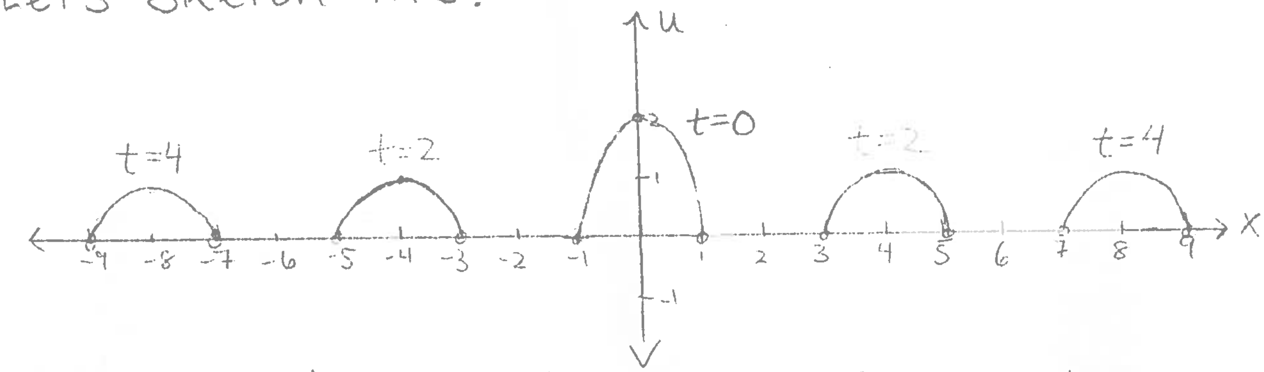
and

$$f(x+8)=\begin{cases} 2(1-(x+8)^2) & \text{if } -1<x+8<1 \\ 0 & \text{else} \end{cases}$$

$$=\begin{cases} 2(1-(x^2+16x+64)) & \text{if } -9<x<-7 \\ 0 & \text{else} \end{cases}$$

$$=\begin{cases} -2(x+7)(x+9) & \text{if } -9<x<-7 \\ 0 & \text{else} \end{cases}$$

Let's sketch this!



*Since we have $\alpha=2$, the waves are moving apart at a speed of 4