Math 319 **Fall 2015**

Sample Final Exam

Note: The final exam is scheduled for **Monday, December 7** at **9:00am** in **Art 214**. You will have three hours to complete the exam and it is worth 40% of your grade. Regardless of your grade going into the final exam, you need to obtain at least 40% on the final exam to pass the course. You will be allowed to use a calculator, but you will not be permitted to bring any notes into the exam. You will be given the useful identities and integrals list on the exam. In order to receive full marks for a solution, you are expected to show all work. The actual final exam will be structured the same way as this sample exam. It will be out of the same number of marks and have the same number of questions. Below are some examples of the style of problems you could see on the exam. These are not the only types of questions I will ask you. The point of this sample exam is to show you approximately how long and challenging the final exam will be. It is worthwhile for you to review your midterms, practice midterms, lecture and tutorial assignments, and the examples we covered in the lecture and tutorial notes, as I could put problems similar to any of these questions on the final exam. You should expect to see approximately three questions related to the material after the second midterm and four questions related to the material from the first two midterms. In this course we have covered Sections 10.1-10.7 and 11.1-11.5 in the textbook. We also covered some additional topics that do not appear in these sections, such as the method of characteristic lines.

Useful Identities:

$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

$2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$

$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$

$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y)$

$\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

Useful Integrals: Let a be a constant, then

$$\int e^x \sin(ax)dx = \frac{e^x}{a^2+1}(\sin(ax) - a\cos(ax)),$$

and

$$\int e^x \cos(ax)dx = \frac{e^x}{a^2+1}(a\sin(ax) + \cos(ax)).$$

[12 pts] **1.** **(a)** Find a formal solution to the following vibrating string problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad 0 < x < 1 \quad \text{and} \quad t > 0,$$

$$u(0,t) = u(1,t) = 0 \quad \text{for} \quad t > 0,$$

$$u(x,0) = 2\sin(\pi x) \quad \text{for} \quad 0 < x < 1,$$

$$u_t(x,0) = 0 \quad \text{for} \quad 0 < x < 1.$$

When you go through and check the cases, cases 1 and 2 give the trivial solution, so **do not** check them. Only check the case where $\lambda > 0$.

(b) Give physical interpretations for the boundary conditions at $x = 0$ and $x = 1$, and the initial conditions at $t = 0$.

[6 pts] **2.** Find the general solution of $3u_x + 2u_y - 2u = 5x$.

[10 pts] **3.** Find a formal solution to the given boundary value problem.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on} \quad 0 < x < \pi \quad \text{and} \quad 0 < y < 1,$$

$$u_x(0,y) = u(\pi,y) = 0 \quad \text{for} \quad 0 < y < 1,$$

$$u(x,0) = 3\cos\left(\frac{1}{2}x\right) \quad \text{an} \quad u(x,1) = \cos\left(\frac{5}{2}x\right) \quad \text{for} \quad 0 < x < \pi.$$

When you go through and check the cases, cases 1 and 2 give the trivial solution, so **do not** check them. Only check the case where $\lambda > 0$.

[7 pts] **4.** Obtain the Fourier sine series for the function

$$f(x) = \begin{cases} -1, & 0 < x < 1, \\ 2, & 1 < x < 3. \end{cases}$$

Discuss the convergence of this series, and show graphically the function represented by the series for all x.

[12 pts] **5.** Consider the following boundary value problem:

$$x^2 y'' + 3xy' + 26y = c + \frac{1}{x},$$

$$y(1) = 0, \quad y(e^{\pi}) = 0.$$

(a) State the Fredholm alternative.

(b) Find the values of c for which the problem has a solution.

[6 pts] **6.** Let L be the differential operator defined by

$$L[y] := A_2(x)y''(x) + A_1(x)y'(x) + A_0(x)y(x).$$

Start with the inner product $(L[u], v)$ and use integration by parts to obtain formulas for the formal adjoint $L^+[y]$ and the bilinear concomitant $P(u, v)$ defined by Green's formula:

$$\int_a^b \left(L[u]v - uL^+[v]\right)\, dx = P(u,v)(x)\Big|_a^b.$$

[7 pts] **7.** Consider the following boundary value problem.

$$y'' - 2y + \lambda y = 0, \quad 0 < x < \pi,$$

$$y(0) = 0, \quad y'(\pi) = 0.$$

(a) Convert this equation into a Sturm-Liouville equation and verify that the problem is regular.
(b) Use the fact that the eigenvalues and eigenfunctions for this boundary value problem are given by

$$\lambda_n = 2 + \left(\frac{2n-1}{2}\right)^2, \quad \text{for} \quad n = 1,\ 2,\ 3, \ldots.$$

and

$$y_n(x) = c_n \sin\left(\frac{2n-1}{2}x\right),$$

to express the function $f(x) = 1$ in an eigenfunction expansion on $0 < x < \pi$.