**a place of mind**
**THE UNIVERSITY OF BRITISH COLUMBIA**

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Instructor: Rebecca Tyson Course: MATH 319
Date: Dec 13th, 2023 Time: noon Duration: 3 hours.
This exam has 8 questions for a total of 60 points.

UBC ID #: ____________________ NAME (print): ________________________________________

Signature: ______________________________________________________________________________

**SPECIAL INSTRUCTIONS**

- **Show and explain all of your work** unless the question directs otherwise. There will be **no marks for answers without supporting work.**
- Simplify all answers.
- The use of a calculator is NOT permitted.
- Answer the questions in the exam booklet provided, except for those problems where you are asked to plot a solution on the figure provided.. If you run out of room, ask for another exam booklet.

**This is an individual exam. You have three hours to complete all of the questions on the exam.**

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 8 | 8 | 6 | 12 | 6 | 6 | 4 | 10 | 60 |
| Score: | | | | | | | | | |

1. Consider the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{-3y}. \tag{1}$$

$\boxed{2}$ (a) Find the equation for the characteristic lines of (1).

$\boxed{2}$ (b) Use a change of variables $v(w, z) = u(x, y)$ to convert the PDE problem in $u(x, y)$ into an ODE problem in $v(z)$.

$\boxed{4}$ (c) Solve the ODE you obtained in part (b). Give the final answer in terms of $x$ and $y$.

$\boxed{8}$ 2. Compute the Fourier cosine series for the function

$$f(x) = \begin{cases} 0, & 0 < x < 1, \\ x - 1, & 1 < x < 2. \end{cases}$$

Discuss the convergence of this series, and show graphically 3 periods of the function represented by the series for all $x$.

3. Your fine arts friend wants to know what all the mathematical symbols in a PDE problem mean. You explain, using the two different PDE problems below.

$\boxed{3}$ (a) Consider the following problem:

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L,\ t > 0 \tag{2a}$$

$$u(0, t) = u(L, t) = 0 \qquad t > 0 \tag{2b}$$

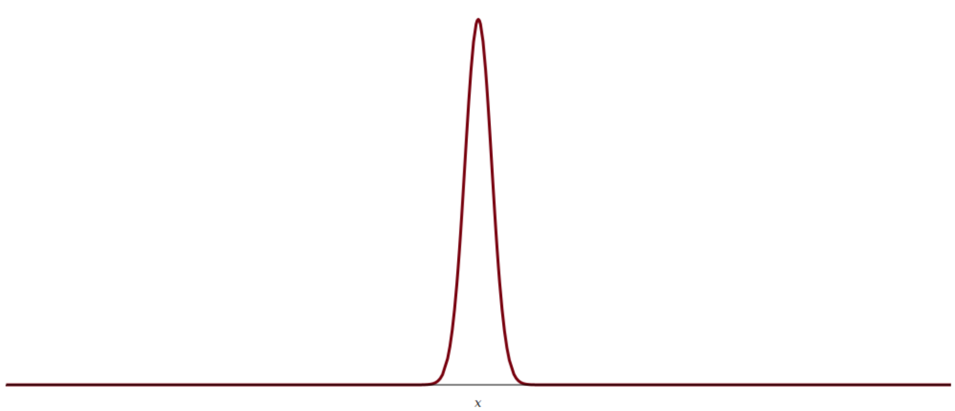
$$u(x, 0) = f(x) \qquad 0 < x < L \tag{2c}$$

where $f(x)$ is shown below.

i. Describe the physical (real world) system that the PDE problem represents. Specify the physical quantity that the variable $u(x, t)$ represents, and how to physically interpret the given boundary conditions and initial condition.

ii. Without solving the PDE problem, write down the expected mathematical form of the solution.

iii. The initial condition $f(x)$ is shown below. Complete the plot by adding the boundaries and two representative solutions at times $0 < t_1 < t_2$. Label each curve with the appropriate time value.



$\boxed{3}$ (b) Consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L,\ t > 0 \tag{3a}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0 \tag{3b}$$

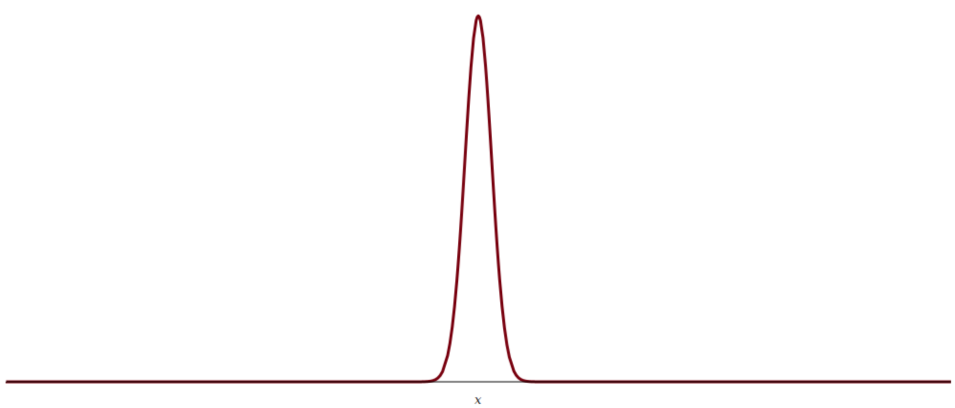
$$u(x,0) = f(x) \qquad 0 < x < L \tag{3c}$$

where $f(x)$ is shown below.

i. Describe the physical (real world) system that the PDE problem represents. Specify the physical quantity that the variable $u(x,t)$ represents, and how to physically interpret the given boundary conditions and initial condition.

ii. Without solving the PDE problem, write down the expected mathematical form of the solution.

iii. The initial condition $f(x)$ is shown below. Complete the plot by adding the boundaries and two representative solutions at steady state.




4. Consider the PDE problem below:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0,$$

$$\frac{\partial u}{\partial x}(0,t) = U_1, \quad u(\pi,t) = U_2, \qquad t > 0,$$

$$u(x,0) = f(x), \qquad 0 < x < \pi.$$

$\boxed{4}$ (a) Derive the steady-state and transient problems that arise from the PDE problem.

$\boxed{2}$ (b) Solve the steady-state problem that arises from the PDE problem.

$\boxed{5}$ (c) Find the formal solution for the transient problem (you may assume that for non-trivial solutions the eigenvalues must be strictly positive).

$\boxed{1}$ (d) Give the formal solution for $u(x,t)$.

5. Consider the initial value problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, && -\infty < x < \infty, \quad t > 0,\\
&u(x,0) = \begin{cases} 2(1-x^2), & \text{if } -1 < x < 1,\\ 0, & \text{else .}\end{cases} && -\infty < x < \infty\\
&\frac{\partial u}{\partial t}(x,0) = 0, && -\infty < x < \infty.
\end{aligned}$$

$\boxed{4}$ (a) Find the solution.

$\boxed{2}$ (b) Plot the solution at $t = 0$, 2, and 4. Make sure that your axes have all appropriate labels and markings.

6. Consider the set of functions
$$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}.$$

$\boxed{4}$ (a) Show that these functions form an orthogonal set on the interval $0 \le x \le L$.

$\boxed{2}$ (b) Normalise the functions so that they form an orthonormal set on $0 \le x \le L$.

$\boxed{4}$ 7. Convert the Bessel equation (below) into the form of a Sturm-Liouville equation. Identify the functions $p(x)$, $q(x)$, and $r(x)$.

$$x^2y'' + xy' + (x^2 - \omega^2)y = 0, \quad \omega \in \mathbb{R}$$

8. Consider the following boundary value problem

$$x^2y'' + 3xy' + 26y = c + \frac{1}{x}, \qquad y(1) = 0, \quad y(e^{\pi}) = 0.$$

$\boxed{5}$ (a) Find the adjoint boundary value problem for the associated homogeneous problem.

$\boxed{5}$ (b) Find the values of $c$ for which the nonhomogeneous boundary value problem has a solution.