Math 319 ~ Sep-Dec 2023
Final Exam ~ Sol'ns

1. Consider the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{-3y}. \qquad (1)$$

$\boxed{2}$ (a) Find the equation for the characteristic lines of (1).

$\boxed{2}$ (b) Use a change of variables $v(w,z) = u(x,y)$ to convert the PDE problem in $u(x,y)$ into an ODE problem in $v(z)$.

$\boxed{4}$ (c) Solve the ODE you obtained in part (b). Give the final answer in terms of x and y.

a) (1) can be written

$$(\hat{i} + \hat{j}) \cdot \vec{\nabla} u = -u + e^{-3y}$$

So we chose new coordinates

$$\begin{cases} w = x - y \\ z = y \end{cases} \iff \begin{cases} x = (w+z) \\ y = z \end{cases} \quad \leftarrow \text{characteristic lines } x - y = k$$

b) and set $v(w,z) = u(x,y)$. Rewriting (1) in terms of $v(w,z)$ we obtain

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + v = e^{-3y} \iff \frac{\partial v}{\partial w}\overbrace{\frac{\partial w}{\partial x}}^{1} + \frac{\partial v}{\partial z}\overbrace{\frac{\partial z}{\partial x}}^{0} + \frac{\partial v}{\partial w}\overbrace{\frac{\partial w}{\partial y}}^{-1} + \frac{\partial v}{\partial z}\overbrace{\frac{\partial z}{\partial y}}^{1} + v = e^{-3z}$$

$$\iff \frac{\partial v}{\partial w} - \frac{\partial v}{\partial w} + \frac{\partial v}{\partial z} + v = e^{-3z}$$

$$\iff \frac{\partial v}{\partial z} + v = e^{-3z} \quad . \; . \; . \quad (1a)$$

c) We can treat (1a) as an ODE in z. We obtain

$$\frac{\partial (e^{z} v)}{\partial z} = e^{-2z} \iff e^{z} v = -\frac{1}{2} e^{-2z} + h(w) \iff v(w,z) = -\frac{1}{2} e^{-3z} + e^{-z} h(w)$$

So we find the solution

$$u(x,t) = -\frac{1}{2} e^{-3y} + e^{-y} h(x-y)$$

Where h is determined by the initial conditions.

So now we finish the integral for a_0:

$$a_n = I - \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}(-1)^n - \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\Bigg|_1^2$$

$$= \cancel{-\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)} + \frac{4}{(n\pi)^2}(-1)^n - \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\underbrace{\sin(n\pi)}_{0} + \cancel{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$

$$= \frac{4}{(n\pi)^2}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]$$

Now we compute the other coefficient

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 0\,dx + \int_1^2 (x-1)\,dx$$

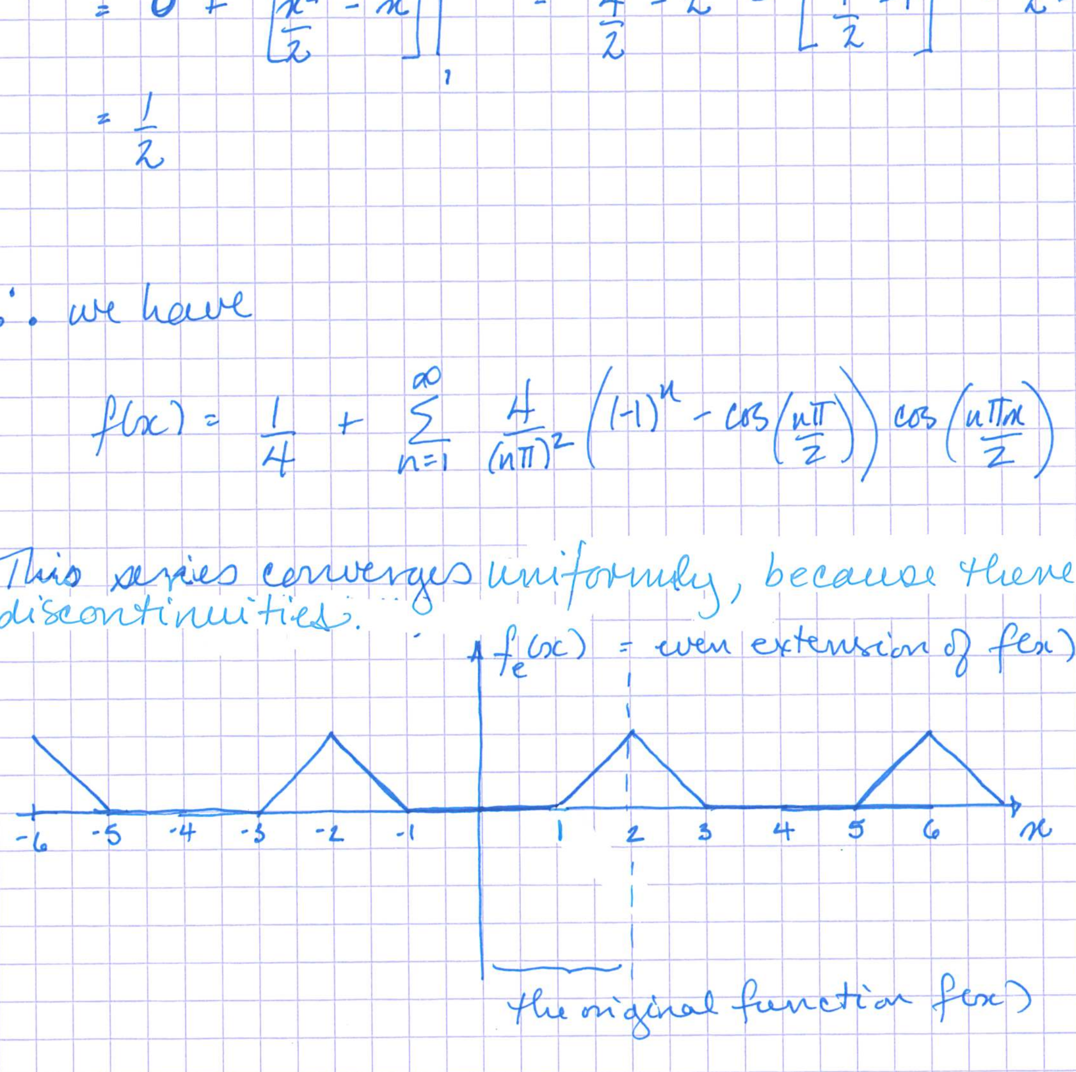
$$= 0 + \left[\frac{x^2}{2} - x\right]_1^2 = \frac{4}{2} - 2 - \left[\frac{1}{2} - 1\right] = 2 - 2 - \frac{1}{2} + 1$$

$$= \frac{1}{2}$$

$\therefore$ we have

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{2}\right)$$

This series converges uniformly, because there are no discontinuities.



$\boxed{8}$ 2. Compute the Fourier cosine series for the function

$$f(x) = \begin{cases} 0, & 0 < x < 1, \\ x - 1, & 1 < x < 2. \end{cases}$$

Discuss the convergence of this series, and show graphically 3 periods of the function represented by the series for all x.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad \text{where } L = 2$$

The coefficients a_n are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 0 \cdot \cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (x-1)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \underbrace{\int_1^2 x\cos\left(\frac{n\pi x}{2}\right)dx}_{I} - \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

To solve I we use integration by parts:

let $u = x$ then $du = dx$

$dv = \cos\left(\frac{n\pi x}{2}\right)dx \quad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

and so

$$I = \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 - \frac{2}{n\pi}\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}2\sin(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2$$

$$= \frac{4}{n\pi}\cdot 0 - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}(-1)^n - \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

3. Your fine arts friend wants to know what all the mathematical symbols in a PDE problem mean. You explain, using the two different PDE problems below.

$\boxed{3}$ (a) Consider the following problem:

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L,\ t > 0 \qquad \text{(2a)}$$

$$u(0,t) = u(L,t) = 0 \qquad t > 0 \qquad \text{(2b)}$$

$$u(x,0) = f(x) \qquad 0 < x < L \qquad \text{(2c)}$$

where $f(x)$ is shown below.

i. Describe the physical (real world) system that the PDE problem represents. Specify the physical quantity that the variable $u(x,t)$ represents, and how to physically interpret the given boundary conditions and initial condition.

Equation (2a) is often called the heat equation because it can be used to describe how the distribution of heat evolves over time. As formulated (2) describes how heat is redistributed in a linear (insulated) rod with ends held at 0°C (or whatever unit heat is measured in), when the initial distribution of heat is $f(x)$.

$u(x,t)$ → the amount of heat in the rod at position x and time t

$u(0,t)$ → the heat in the rod at the point $x=0$

$u(L,t)$ → " " " " " " " " $x=L$

$u(x,0)$ → the initial distribution of heat in the rod (at $t=0$)

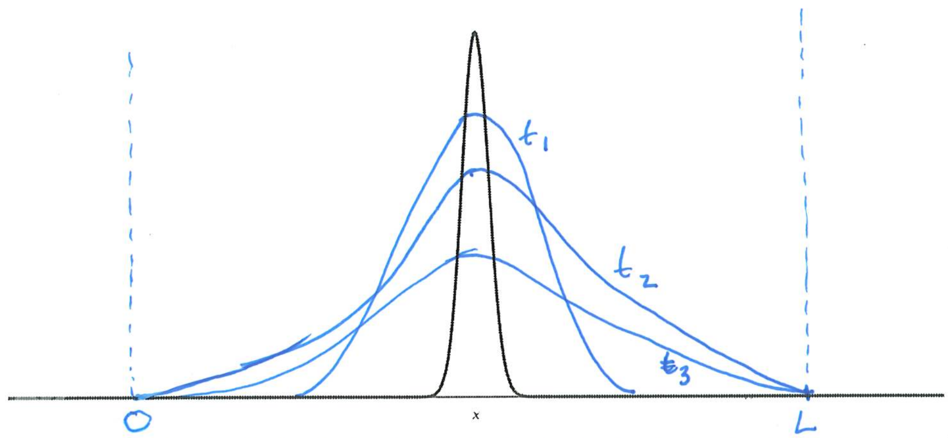
ii. Without solving the PDE problem, write down the expected mathematical form of the solution.

Since we have $u(0,t)=0$ we expect the spatial eigenfunctions to be sine functions. We also know that the time component of the solution will be a decaying exponential, since heat dissipates over time. So we expect

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

where the coefficients b_n are given by the Fourier expansion of $f(x)$.

iii. The initial condition $f(x)$ is shown below. Complete the plot by adding the boundaries and two representative solutions at times $0 < t_1 < t_2$. Label each curve with the appropriate time value.



$\boxed{3}$ (b) Consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L,\ t > 0 \qquad (3a)$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0 \qquad (3b)$$

$$u(x,0) = f(x) \qquad 0 < x < L \qquad (3c)$$

where $f(x)$ is shown below.

i. Describe the physical (real world) system that the PDE problem represents. Specify the physical quantity that the variable $u(x,t)$ represents, and how to physically interpret the given boundary conditions and initial condition.

Equation (3a) is often called the wave equation because it describes the movement of waves in an incompressible medium like water. Since this is a problem in one space dimension, we can think of it as modelling the movement of a vibrating string.

$u(x,t)$ = the position (height) of the string (relative to its rest position at 0) at position x and time t

$\frac{\partial u}{\partial x}(0,t) = 0$ → this BC means that the $x=0$ end of the string is free to move, + that waves are symmetric about $x=0$

$\frac{\partial u}{\partial x}(L,t) = 0$ → this BC means that the $x=L$ end of the string is free to move, + that waves are symmetric about $x=L$.

$u(x,0) = f(x)$ → this is the initial ($t=0$) position of the string

ii. Without solving the PDE problem, write down the expected mathematical form of the solution.

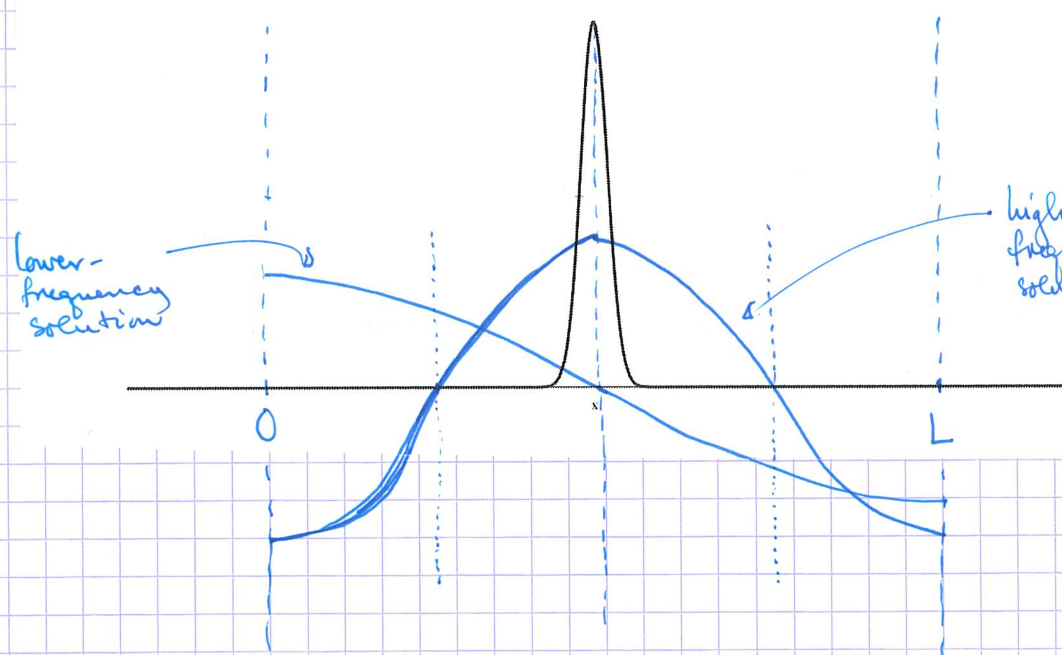
Because of the Neumann (0 derivative) BCs, we expect the spatial eigenfunctions to be cosines. The second derivative in time means that the time-component of the solution is also sinusoidal.

We expect solutions of the form

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(c\frac{n\pi t}{L}\right)\right) \cos\left(\frac{n\pi x}{L}\right)$$

where the A & B coefficients are determined by the Fourier expansions of the initial conditions.

iii. The initial condition $f(x)$ is shown below. Complete the plot by adding the boundaries and two representative solutions at steady state.



4. Consider the PDE problem below:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0,$$

$$\frac{\partial u}{\partial x}(0,t) = U_1, \quad u(\pi,t) = U_2, \qquad t > 0,$$

$$u(x,0) = f(x), \qquad 0 < x < \pi.$$

$\boxed{4}$ (a) Derive the steady-state and transient problems that arise from the PDE problem.

$\boxed{2}$ (b) Solve the steady-state problem that arises from the PDE problem.

$\boxed{5}$ (c) Find the formal solution for the transient problem (you may assume that for non-trivial solutions the eigenvalues must be strictly positive).

$\boxed{1}$ (d) Give the formal solution for $u(x,t)$.

a) Let $u(x,t) = \underbrace{v(x)}_{\text{steady-state}} + \underbrace{w(x,t)}_{\text{transient}}$. Then the PDE problem becomes

$$(2) \cdots \begin{cases} \dfrac{\partial w}{\partial t} = \beta \dfrac{\partial^2 w}{\partial x^2} + \beta v'' & 0 < x < \pi,\ t > 0 \\ \dfrac{\partial w}{\partial x}(0,t) + v'(0) = U_1 & t > 0 \\ w(\pi,t) + v(\pi) = U_2 & t > 0 \\ w(x,0) + v(x) = f(x) & 0 < x < \pi \end{cases}$$

Since (2) must be true $\forall t$, we let $t \to \infty$. Since $w(x,t)$ is transient

$$\lim_{t\to\infty} w(x,t) = 0$$

and so (2) gives rise to the steady-state problem

$$v'' = 0, \quad v'(0) = U_1, \quad v(\pi) = U_2 \qquad \ldots (3)$$

and the transient problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = \beta \dfrac{\partial^2 w}{\partial x^2} & 0 < x < \pi,\ t > 0 \\ \dfrac{\partial w}{\partial x}(0,t) = 0, \quad w(\pi,t) = 0 & t > 0 \\ w(x,0) = f(x) - v(x) = F(x) & 0 < x < \pi \end{cases} \qquad \cdots (4)$$

(b) The solution to (3) is found by integrating the ODE

$$v'' = 0 \iff v = Ax + B$$

and applying the BCs:

$$\begin{cases} v'(0) = U_1 \\ v(\pi) = U_2 \end{cases} \iff \begin{cases} A = U_1 \\ U_1\pi + B = U_2 \end{cases} \iff \begin{cases} A = U_1 \\ B = U_2 - U_1\pi \end{cases}$$

So $v(x) = U_1 x + (U_2 - U_1\pi)$

(c) To solve (4) we assume that $w(x,t) = X(x)T(t)$. Plugging this formulation into (4) we obtain

$$XT' = \beta X''T \iff \frac{1}{\beta}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

So

$$\text{(A)} \begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X(\pi) = 0 \end{cases} \qquad \text{(B)} \begin{cases} T' + \beta\lambda T = 0 \end{cases}$$

Assuming $\lambda = w^2 > 0$, the solutions to (A) are

$$X(x) = a\cos(wx) + b\sin(wx)$$

Applying the BCs we obtain

$$X'(0) = -wa\sin(0) + wb\cos(0) = wb\cos(0) = wb = 0$$

so $b = 0$ and

$$X(\pi) = 0 \iff a\cos(w\pi) = 0 \iff w\pi = (2n-1)\frac{\pi}{2}, \ n \in \mathbb{N}$$

$$\iff w = \frac{(2n-1)}{2}, \ n \in \mathbb{N}$$

$\therefore$ the spatial eigenfunctions are

$$X_n(x) = \cos\left(\frac{(2n-1)}{2}x\right)$$

The solutions to (B) are

$$T' + \frac{\beta}{4}(2n-1)^2 T = 0 \iff T = Ce^{-\frac{\beta}{4}(2n-1)^2 t}$$

Putting the two solutions together we have

$$w(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)}{2}x\right) e^{-\frac{\beta}{4}(2n-1)^2 t}$$

where the coefficients a_n are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} F(x)\cos\left(\frac{(2n-1)}{2}x\right)dx, \quad n \in \mathbb{N} \quad \ldots \quad (5)$$

(d) Finally, we obtain

$$u(x,t) = w(x,t) + v(x)$$

$$= U_1 x + (U_2 - U_1\pi) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)}{2}x\right) e^{-\frac{\beta}{4}(2n-1)^2 t}$$

where the coefficients a_n are given by (5) & $F(x)$ is defined in (4).

5. Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = \begin{cases} 2(1-x^2), & \text{if } -1 < x < 1, \\ 0, & \text{else .} \end{cases} \qquad -\infty < x < \infty$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad -\infty < x < \infty.$$

$\boxed{4}$ (a) Find the solution.

$\boxed{2}$ (b) Plot the solution at $t = 0$, 2, and 4. Make sure that your axes have all appropriate labels and markings.

(a) The general solution is $u(x,t) = A(x+2t) + B(x-2t)$. We need to choose the functions A and B so that the ICs are satisfied.

$$\begin{cases} u(x,0) = A(x) + B(x) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = 2A'(x) - 2B'(x) = 0 \end{cases} \quad \ldots \quad (6)$$

Solving (6b) we have

$$2A'(x) - 2B'(x) = 0 \iff A'(x) = B'(x) \iff A(x) = B(x) + C$$

where C is a constant.

We thus arrive at the following system of equations:

$$\begin{cases} A(x) + B(x) = f(x) \\ A(x) - B(x) = C \end{cases} \iff \begin{cases} 2A(x) = f(x) + C \\ 2B(x) = f(x) - C \end{cases}$$

$$\iff \begin{cases} A(x) = \frac{1}{2} f(x) + \frac{C}{2} \\ B(x) = \frac{1}{2} f(x) - \frac{C}{2} \end{cases}$$

Then

$$u(x,t) = A(x+2t) + B(x-2t)$$

$$= \frac{1}{2} f(x+2t) + \frac{C}{2} + \frac{1}{2} f(x-2t) - \frac{C}{2}$$

$$= \frac{1}{2} f(x+2t) + \frac{1}{2} f(x-2t)$$

where

$$f(x+2t) = \begin{cases} 2\left(1 - (x+2t)^2\right) & \text{if } -1 < x+2t < 1 \\ 0 & \text{else} \end{cases}$$

$$f(x-2t) = \begin{cases} 2\left(1 - (x-2t)^2\right) & \text{if } -1 < x-2t < 1 \\ 0 & \text{else} \end{cases}$$

(b) At $t = 0$ we have $u(x,0) = f(x)$

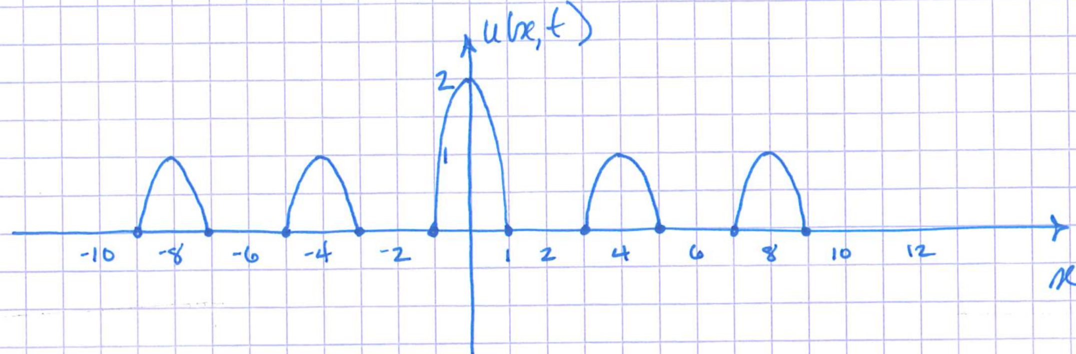
At $t = 2$ we have

$$u(x,2) = \frac{1}{2}\left[\underset{\substack{\uparrow \\ \text{shifted} \\ \text{left } 4}}{f(x+4)} + \underset{\substack{\uparrow \\ \text{shifted} \\ \text{right } 4}}{f(x-4)}\right]$$

At $t = 4$ we have

$$u(x,4) = \frac{1}{2}\left[f(x+8) + f(x-8)\right]$$

shifted left 8   shifted right 8



6. Consider the set of functions

$$\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}.$$

$\boxed{4}$ (a) Show that these functions form an orthogonal set on the interval $0 \le x \le L$.

$\boxed{2}$ (b) Normalise the functions so that they form an orthonormal set on $0 \le x \le L$.

a) We need to show that

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \neq 0 \qquad \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = 0, \quad n \neq m$$

Consider first the second integral. From the formula sheet, we know that

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)$$

$\therefore$ the second integral evaluates to

$$I = \int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right] dx$$

So

$$II = \frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\Bigg|_0^L - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\Bigg|_0^L$$

$$= \frac{L}{(n-m)\pi}\sin\left((n-m)\pi\right) - 0 - \frac{L}{(n+m)\pi}\sin\left((n+m)\pi\right) + 0$$

But $n, m \in \mathbb{N}$ and $n \neq m$ so $n-m \in \mathbb{Z}\setminus\{0\}$ and $n+m \in \mathbb{N}$, which means that the two sin terms above evaluate to 0.

$\therefore II = 0$ as required.

Now for the first integral

$$I = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2n\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$= \frac{1}{2}\left[L - 0 - \frac{L}{2n\pi}\sin(2n\pi) + 0\right] = \frac{L}{2} \neq 0$$

as required. $\therefore$ the given set of functions is orthogonal.

(b) To make the functions ortho<u>normal</u>, we divide by $\sqrt{L/2}$. The new set is

$$\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$$

We can check by showing that $\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)dx = 1$.

Using I from above, we can immediately see that this result holds.

$\boxed{4}$ 7. Convert the Bessel equation (below) into the form of a Sturm-Liouville equation. Identify the functions $p(x)$, $q(x)$, and $r(x)$.

$$x^2y'' + xy' + (x^2 - \omega^2)y = 0, \quad \omega \in \mathbb{R}$$

$A_2(x) = x^2$, $A_1(x) = x$

Compute the integrating factor

$$\mu(x) = \frac{1}{A_2(x)} \exp\left(\int \frac{A_1(x)}{A_2(x)}dx\right) = \frac{1}{x^2} e^{\int \frac{x}{x^2} dx}$$

$$= \frac{1}{x^2} e^{\int \frac{1}{x} dx} = \frac{1}{x^2} e^{\ln(x)} = \frac{1}{x^2} x = \frac{1}{x}$$

Multiplying the Bessel equation by $\mu(x) = \frac{1}{x}$ we obtain

$$xy'' + y' + xy - \frac{\omega^2}{x}y = 0 \text{ iff}$$

$$(xy')' + xy - \frac{\omega^2}{x}y = 0$$

This ODE is in S-L form where

$p(x) = x$, $q(x) = x$, and $r(x) = \frac{1}{x}$

8. Consider the following boundary value problem

$$x^2y'' + 3xy' + 26y = c + \frac{1}{x}, \qquad y(1) = 0, \quad y(e^\pi) = 0.$$

$\boxed{5}$ (a) Find the adjoint boundary value problem for the associated homogeneous problem.

$\boxed{5}$ (b) Find the values of c for which the nonhomogeneous boundary value problem has a solution.

(a) We are given $L[y]: x^2y'' + 3xy' + 26y$
$B[y]: y(1) = y(e^\pi) = 0$

with $h(x) = c + \frac{1}{x}$

To find the adjoint BVP, observe that $A_2 = x^2$, $A_1 = 3x$, $A_0 = 26$.

So $L^+[y] = (A_2 y)'' - (A_1 y)' + (A_0 y)$

$= (x^2 y)'' - (3xy)' + 26y$

$= (2xy + x^2 y')' - (3xy' + 3y) + 26y$

$= 2y + 2xy' + 2xy' + x^2 y'' - 3y - 3xy' + 26y$

$= x^2 y'' + xy' + 25y$

To find the adjoint BCs, we need

$P(u,v)\Big|_1^{e^\pi} = 0 \quad \forall\ u \in D(L)$ and $v \in D(L^+)$

we compute

$P(u,v) = u A_1 v - u(A_2 v)' + u' A_2 v$

$= u(3x)v - u(x^2 v)' + u' x^2 v$

$= 3xuv - u(2xv + x^2 v') + u' x^2 v$

$= xuv - x^2 uv' + x^2 u'v$

Now set

$P(u,v)\Big|_1^{e^\pi} = 0 \iff e^\pi \cancel{u(e^\pi)}^{\,0} v(e^\pi) - e^{2\pi} \cancel{u(e^\pi)}^{\,0} v'(e^\pi) + e^{2\pi} u'(e^\pi) v(e^\pi)$

$- \left(\cancel{u(1)}^{\,0} v(1) - \cancel{u(1)}^{\,0} v'(1) + u'(1) v(1) \right) = 0$

$\iff e^{2\pi} u'(e^\pi) v(e^\pi) - u'(1) v(1) = 0$

$\because u'(e^\pi)$ and $u'(1)$ are arbitrary, we must have

$B(L^+)$: $v(1) = v(e^\pi) = 0$

Thus, the adjoint BVP is

$L^+[y]$: $x^2 y'' + xy' + 25y = 0$

$B^+[y]$: $y(1) = y(e^\pi) = 0$

(b) To solve the adjoint BVP we set $y = x^r$. Plugging this soln into $L^+[y]$ we obtain

$$x^2 r(r-1)x^{r-2} + x r x^{r-1} + 25x^r = 0 \Leftrightarrow (IV)$$

$$(IV) \Leftrightarrow r(r-1) + r + 25 = 0 \Leftrightarrow r^2 - r + r + 25 = 0 \Leftrightarrow r^2 + 25 = 0$$

$$\Leftrightarrow r = \pm 5i$$

$\therefore\ y = C_1 \cos(5\ln(x)) + C_2 \sin(5\ln(x))$

Now apply $B^+[y]$:

$$y(1) = 0 \Leftrightarrow C_1 \cos(0) + C_2 \sin(0) = C_1 = 0$$

$$y(e^{\pi}) = 0 \Leftrightarrow C_2 \sin(5\pi) = 0 \quad \text{true } \forall C_2$$

$\therefore\ y(x) = C_2 \sin(5\ln(x))$ where C_2 is arbitrary.

By the Fredholm Alternative, the nonhomogeneous problem has a solution iff

$$\int_1^{e^{\pi}} \left(c + \frac{1}{x}\right) \sin(5\ln(x))\, dx = 0 \Leftrightarrow (IV)$$

$$(IV) \Leftrightarrow \int_1^{e^{\pi}} c \sin(5\ln(x))\, dx + \int_1^{e^{\pi}} \frac{1}{x} \sin(5\ln(x))\, dx = 0$$

let $\begin{cases} u = \ln x \Rightarrow x = e^u \\ du = \frac{1}{x} dx \Rightarrow dx = e^u du \end{cases}$ then $\begin{cases} x = 1 \Rightarrow u = 0 \\ x = e^{\pi} \Rightarrow u = \pi \end{cases}$

and we can rewrite (IV):

$$(IV) \Leftrightarrow \underbrace{c \int_0^{\pi} \sin(5u)\, e^u\, du}_{(A)} + \underbrace{\int_0^{\pi} \sin(5u)\, du}_{(B)} = 0$$

From the formula sheet we have

$$A = \frac{e^u}{5^2+1}\left(\sin(5u) - 5\cos(5u)\right)\Big|_0^{\pi}$$

$$= \frac{1}{26}\left[e^{\pi}\sin(5\pi) - 5e^{\pi}\cos(5\pi) - (0 - 5\cdot 1)\right]$$

$$= \frac{5}{26}\left(e^{\pi}+1\right)$$

The second integral is straightforward

$$B = -\frac{\cos(5u)}{5}\Big|_0^{\pi} = -\frac{1}{5}\left(\cos(5\pi) - 1\right) = \frac{2}{5}$$

$$\therefore \text{IV} \Leftrightarrow \frac{5c}{26}\left(e^{\pi}+1\right) + \frac{2}{5} = 0 \Leftrightarrow c = -\frac{2}{5}\,\frac{26}{5(e^{\pi}+1)}$$

$$\Leftrightarrow c = \frac{-52}{25(e^{\pi}+1)} \quad \cdots\cdots \quad (7)$$

Thus, the given BVP has a solution if c satisfies (7).