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Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson   Course: MATH 319
Date: Oct 4th, 2023   Time: 1:00pm Duration: 45 minutes.
This exam has 5 questions for a total of 48 points.

UBC ID #: ____________________ NAME (print): ____

Signature: ____

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 45 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Note: The table of marks appears on the second last page of the exam.

10 1. Consider the PDE

$$2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + 5u = xy. \tag{1}$$

Show, using an appropriate transformation of variables, that the PDE (1) in $u(x, y)$ can be converted to a PDE in $v(w, z)$ with only one derivative (in a single variable), where (w, z) are the new independent variables. Show all steps in your work. Do NOT solve the ODE.

7 2. Apply the technique of "separation of variables" to the PDE

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \alpha^2 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right). \tag{2}$$

If the technique works, give the two ODEs that result. If the technique does not work, explain why. Note that D and α are constants.

11 3. The heat flow problem

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \quad t > 0 \tag{3a}$$

$$u(0,t) = u(2,t) = 0, \qquad t > 0 \tag{3b}$$

$$u(x,0) = f(x), \qquad 0 < x < 2, \tag{3c}$$

where

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{2}\right), \tag{4}$$

can be solved using separation of variables. Assuming that $u(x,t) = X(x)T(t)$, we find that the PDE and boundary conditions can be rewritten as a pair of ODEs:

$$\text{Problem A}: \qquad X'' + \lambda X = 0, \qquad X(0) = X(2) = 0,$$

$$\text{Problem B}: \qquad T' + 5\lambda T = 0.$$

If $\lambda \leq 0$, Problem A only admits trivial solutions. Finish solving for the formal solution of the heat flow problem (3) with (4), assuming that $\lambda = \omega^2$ (i.e. start partway through step 3).

workspace for problem #3

13 4. Consider the function

$$f(x) = e^x, \qquad 0 < x < \frac{\pi}{3}.$$

(a) On the axes below, and using two different colours or line types, sketch

i. $f(x)$, and

ii. three periods of the function $g(x)$ to which the Fourier cosine series of $f(x)$ converges.

Label all important axis values (only a few have been provided).

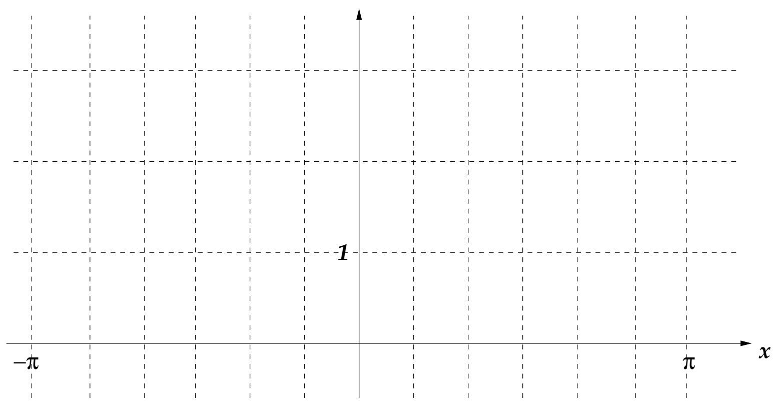

Figure 1: Plot of $f(x)$ and for question a.

(b) Will convergence be uniform or pointwise? (Circle the correct response.) Explain.

(c) Find the Fourier cosine series for $f(x)$. *You may find it useful to consult the integral table provided on the inside back page of the test.*

workspace for question #4

5. Consider the IBVP

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\ t > 0, \tag{6}$$
$$u(0,t) = U_1,\ u(L,t) = U_2, \qquad t > 0, \tag{7}$$
$$u(x,0) = f(x), \qquad 0 < x < L, \tag{8}$$

where $L > 0$ and $\gamma > 0$.

2 (a) Explain why the solution is composed of a steady state portion $v(x)$ and a transient portion $w(x,t)$.

3 (b) Derive the steady state problem and find the solution.

Question #5 continues on the next page.

Question #5 continued.

2 (c) Set up the IBVP for the transient portion.

Possibly Useful Information

Some integrals you may find useful:

$$\int x\sin(\rho x)\,dx = -\frac{x}{\rho}\cos(\rho x) + \frac{1}{\rho^2}\sin(\rho x)$$
$$\int x\cos(\rho x)\,dx = \frac{x}{\rho}\sin(\rho x) + \frac{1}{\rho^2}\cos(\rho x)$$
$$\int e^x\sin(\rho x)\,dx = \frac{e^x}{\rho^2+1}\left[\sin(\rho x) - \rho\cos(\rho x)\right]$$
$$\int e^x\cos(\rho x)\,dx = \frac{e^x}{\rho^2+1}\left[\rho\sin(\rho x) + \cos(\rho x)\right]$$
$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$
$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$$

Question:	1	2	3	4	5	Total
Points:	10	7	11	13	7	48
Score:						