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THE UNIVERSITY OF BRITISH COLUMBIA

IRVING K. BARBER SCHOOL
OF ARTS AND SCIENCES
UBC OKANAGAN

Instructor: Rebecca Tyson Course: MATH 319
Date: Oct 4th, 2023 Time: 1:00pm Duration: 45 minutes.
This exam has 5 questions for a total of 48 points.

UBC ID #: ________________ NAME (print): Solutions

Signature: ________________________________

SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

This is a two-stage exam. You have 45 minutes to complete the exam individually, then you will hand in the tests and join your group to redo the test as a group in the remaining 35 minutes.

Note: The table of marks appears on the second last page of the exam.

6 $\boxed{\cancel{10}}$ 1. Consider the PDE

$$2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + 5u = xy. \quad (1)$$

Show, using an appropriate transformation of variables, that the PDE (1) in $u(x, y)$ can be converted to a PDE in $v(w, z)$ with only one derivative (in a single variable), where (w, z) are the new independent variables. Show all steps in your work. Do NOT solve the ODE.

Let $\begin{cases} w = -1x - 2y \\ z = y \end{cases}$ and $v(w,z) = u(x,y)$

Then

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} = -\frac{\partial v}{\partial w} \quad \ldots\ (1a) \\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} = -2\frac{\partial v}{\partial w} + \frac{\partial v}{\partial z} \ldots\ (1b) \end{cases}$$

Substituting (1a) and (1b) into (1) we obtain

$$2\left(-\frac{\partial v}{\partial w}\right) - \left(-2\frac{\partial v}{\partial w} + \frac{\partial v}{\partial z}\right) + 5v = (-w-2z)z \iff$$

$$\iff -\frac{\partial v}{\partial z} + 5v = -(w+2z)z$$

This last eqn is a PDE with a derivative with respect to only one variable (z). The other variable can therefore be considered constant from the point of view of integration, and the eqn treated like an ODE.

7 $\cancel{11}$ 3. The heat flow problem

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \quad t > 0 \tag{3a}$$
$$u(0,t) = u(2,t) = 0, \qquad t > 0 \tag{3b}$$
$$u(x,0) = f(x), \qquad 0 < x < 2, \tag{3c}$$

where

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{2}\right), \tag{4}$$

can be solved using separation of variables. Assuming that $u(x,t) = X(x)T(t)$, we find that the PDE and boundary conditions can be rewritten as a pair of ODEs:

$$\text{Problem A}: \quad X'' + \lambda X = 0, \qquad X(0) = X(2) = 0,$$
$$\text{Problem B}: \quad T' + 5\lambda T = 0.$$

If $\lambda \leq 0$, Problem A only admits trivial solutions. Finish solving for the formal solution of the heat flow problem (3) with (4), assuming that $\lambda = \omega^2$ (i.e. start partway through step 3).

(A) If $\lambda = \omega^2 > 0$, then (A) gives

$X'' + \omega^2 X = 0 \iff X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$

Now apply the BCs:

$X(0) = 0 \iff c_1 = 0$; $X(2) = 0 \iff c_2 \sin(2\omega) = 0 \Rightarrow 2\omega = n\pi \iff \omega = \frac{n\pi}{2}$, $n \in \mathbb{N}$

(or $c_2 = 0$, which is not interesting)

$\therefore X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$, $n \in \mathbb{N}$ with $\lambda = \omega^2 = \left(\frac{n\pi}{2}\right)^2$

(B) Solving for T we have

$T' + 5\omega^2 T = 0 \iff T = c_3 e^{-5\omega^2 t}$ so $T_n(t) = e^{-5\left(\frac{n\pi}{2}\right)^2 t}$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) e^{-5\left(\frac{n\pi}{2}\right)^2 t}$$

$\boxed{7}$ 2. Apply the technique of "separation of variables" to the PDE

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \alpha^2 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right). \tag{2}$$

If the technique works, give the two ODEs that result. If the technique does not work, explain why. Note that D and α are constants.

Let $u(x,t) = X(x)\,T(t)$. Then (2) becomes

$$XT' = DX''T - \alpha^2 \frac{\partial}{\partial x}(xX'T) = DX''T - \alpha^2(X'T + xX''T) \iff /1/$$

$$/1/ \iff \frac{XT'}{XT} = \frac{DX''T}{XT} - \alpha^2\frac{X'T}{XT} - \alpha^2 x\frac{X''T}{XT}$$

$$\iff \frac{T'}{T} = D\frac{X''}{X} - \alpha^2\frac{X'}{X} - \alpha^2 x\frac{X''}{X} \quad \ldots \quad (2a)$$

We see that the LHS of (2a) depends only on t, + the RHS of (2a) depends only on x. So we can conclude that each must be equal to a constant, which we call $-\lambda$. We thus arrive at two ODEs:

$$D\frac{X''}{X} - \alpha^2\frac{X'}{X} - \alpha^2 x\frac{X''}{X} = -\lambda \iff (D - \alpha^2 x)X'' - \alpha^2 X' + \lambda X = 0 \ldots (2b)$$

$$\frac{T'}{T} = -\lambda \iff T' + \lambda T = 0 \quad \ldots (2c)$$

Eqns (2b) + (2c) are the ODEs.

workspace for problem #3

Now apply the IC:

$$u(x,0)=f(x) \iff \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) = f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{2}\right)$$

$\therefore$ $b_n = \frac{1}{n^2}$ and we have

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{2}\right) e^{-5\left(\frac{n\pi}{2}\right)^2 t}}$$

13 4. Consider the function

$$f(x) = e^x, \qquad 0 < x < \frac{\pi}{3}.$$

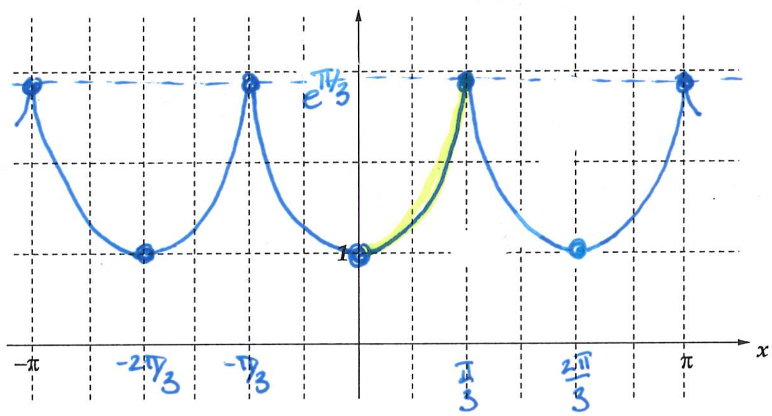
note that $e^{\pi/3} \simeq e^1 \simeq 2.7$

(a) On the axes below, and using two different colours or line types, sketch

i. $f(x)$, and

ii. three periods of the function $g(x)$ to which the Fourier cosine series of $f(x)$ converges.

Label all important axis values (only a few have been provided).



original function $f(x)$

Figure 1: Plot of $f(x)$ and for question a.

(b) Will convergence be uniform or pointwise? (Circle the correct response.) Explain.

The function $g(x)$ is continuous, and so convergence is uniform

(c) Find the Fourier cosine series for $f(x)$. *You may find it useful to consult the integral table provided on the inside back page of the test.*

$$a_0 = \frac{1}{\pi/3}\int_{-\pi/3}^{\pi/3} g(x)\,dx = \frac{2}{\pi/3}\int_0^{\pi/3} f(x)\,dx = \frac{6}{\pi}\int_0^{\pi/3} e^x\,dx = \frac{6}{\pi}\, e^x \Big|_0^{\pi/3}$$

even, so

$$= \frac{6}{\pi}\left(e^{\pi/3} - 1\right)$$

workspace for question #4

$$a_n = \frac{1}{\pi/3}\int_{-\pi/3}^{\pi/3} g(x)\cos\left(\frac{n\pi x}{\pi/3}\right)dx = \frac{2}{\pi/3}\int_0^{\pi/3} e^x \cos\left(\frac{n\pi x}{\pi/3}\right)dx = \frac{6}{\pi}\int_0^{\pi/3} e^x\cos(3nx)\,dx$$

even · even = even, so

$$= \frac{6}{\pi}\left[\frac{e^x}{(3n)^2+1}\left(3n\sin(3nx)+\cos(3nx)\right)\right]_0^{\pi/3}$$ (using the Table on p. 10)

$$= \frac{6}{\pi}\,\frac{1}{9n^2+1}\left[e^{\pi/3}\left(3n\underbrace{\sin(n\pi)}_{0}+\underbrace{\cos(3n\tfrac{\pi}{3})}_{(-1)^n}\right) - 3n\underbrace{\sin(0)}_{0} - \underbrace{\cos(0)}_{1}\right]$$

$$= \frac{6}{\pi}\frac{1}{(9n^2+1)}\left(e^{\pi/3}(-1)^n - 1\right)$$

$$\therefore\ \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{\pi/3}\right) = \frac{3}{\pi}\left(e^{\pi/3}-1\right) + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{e^{\pi/3}(-1)^n - 1}{9n^2+1}\cos(3nx)}$$ on $0 < x < \pi/3$

$= g(x)$ on $x \in \mathbb{R}$

5. Consider the IBVP

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\ t > 0, \tag{6}$$
$$u(0,t) = U_1,\ u(L,t) = U_2, \qquad t > 0, \tag{7}$$
$$u(x,0) = f(x), \qquad 0 < x < L, \tag{8}$$

where $L > 0$ and $\gamma > 0$.

$\boxed{2}$ (a) Explain why the solution is composed of a steady state portion $v(x)$ and a transient portion $w(x,t)$.

The nonhomogeneous boundary conditions suggest there will be a non-zero steady state solution.

We also expect the solution to include a transient portion b/c (6) is a heat equation which should dissipate the IC given by (8).

4 $\boxed{\cancel{8}}$ (b) Derive the steady state problem and find the solution.

Let $u(x,t) = v(x) + w(x,t)$. Then (6) becomes

$$\frac{\partial(v+w)}{\partial t} = \gamma \frac{\partial^2 (v+w)}{\partial x^2} \iff \frac{\partial w}{\partial t} = \gamma\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right] \quad \text{.... (6a)}$$

$\because$ $w(x,t)$ is a transient solution, we expect that $\lim\limits_{t\to\infty} w(x,t) \equiv 0$.

So the steady-state eqn is ....(6b)

$$\gamma \frac{\partial^2 v}{\partial x^2} = 0 \iff \frac{\partial^2 v}{\partial x^2} = 0 \quad \text{or} \quad v'' = 0 \iff v = ax + b$$

Now apply (7) in the limit as $t \to \infty$:

$$\lim_{t\to\infty} u(0,t) = U_1 \iff v(0) = U_1 \iff b = U_1$$
$$\lim_{t\to\infty} u(L,t) = U_2 \iff v(L) = U_2 \iff aL + U_1 = U_2 \iff a = \frac{U_2 - U_1}{L}$$

$$\therefore \boxed{v(x) = \frac{(U_2 - U_1)}{L} x + U_1}$$

Question #5 continues on the next page.

Question #5 continued.

3 $\cancel{\boxed{2}}$ (c) Set up the IBVP for the transient portion.

Now, plugging (6b) into (6a) we have

$$\frac{\partial w}{\partial t} = \gamma \frac{\partial^2 w}{\partial x^2} \qquad 0 < x < L,\ t > 0 \qquad \ldots \quad (6c)$$

The new BCs are

$$u(0,t) = U_1 \iff v(0) + w(0,t) = U_1 \iff U_1 + w(0,t) = U_1 \iff w(0,t) = 0$$
$$u(L,t) = U_2 \iff v(L) + w(L,t) = U_2 \iff U_2 + w(L,t) = U_2 \iff w(L,t) = 0$$

and the new IC is

$$u(x,0) = f(x) \iff v(x) + w(x,0) = f(x)$$
$$\iff \frac{(U_2-U_1)}{L}x + U_1 + w(x,0) = f(x)$$
$$\iff w(x,0) = f(x) - \left[\frac{(U_2-U_1)}{L}x + U_1\right]$$

Putting these together, we have the IBVP in $w(x,t)$:

$$\boxed{\begin{cases} \dfrac{\partial w}{\partial t} = \gamma \dfrac{\partial^2 w}{\partial x^2} & 0 < x < L,\ t > 0 \\ w(0,t) = w(L,t) = 0 & t > 0 \\ w(x,0) = f(x) - \left[\dfrac{(U_2-U_1)}{L}x + U_1\right] & 0 < x < L \end{cases}}$$

Possibly Useful Information

Some integrals you may find useful:

$$\int x\sin(\rho x)\,dx = -\frac{x}{\rho}\cos(\rho x) + \frac{1}{\rho^2}\sin(\rho x)$$

$$\int x\cos(\rho x)\,dx = \frac{x}{\rho}\sin(\rho x) + \frac{1}{\rho^2}\cos(\rho x)$$

$$\int e^x\sin(\rho x)\,dx = \frac{e^x}{\rho^2+1}\left[\sin(\rho x) - \rho\cos(\rho x)\right]$$

$$\int e^x\cos(\rho x)\,dx = \frac{e^x}{\rho^2+1}\left[\rho\sin(\rho x) + \cos(\rho x)\right]$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax}$$

Question:	1	2	3	4	5	Total
Points:	10	7	11	13	7	48
Score:						