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THE UNIVERSITY OF BRITISH COLUMBIA

Irving K. Barber School
of Arts and Sciences
UBC Okanagan

Instructor: Rebecca Tyson Course: MATH 319
Date: Nov 1st, 2023 Time: 12:30pm Duration: 50 minutes.
This exam has 4 questions for a total of 29 points.

UBC ID #: ________________ NAME (print): ______________________________

Signature: Solutions

## SPECIAL INSTRUCTIONS

- Show and explain all of your work unless the question directs otherwise. Simplify all answers.
- The use of a calculator is permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, ask for extra paper.

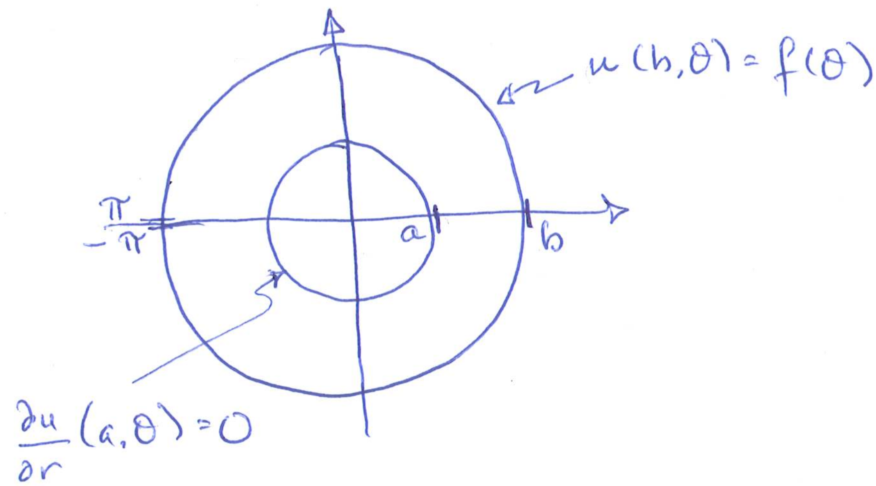
This is the individual portion of a two-stage exam.

1. Consider the PDE problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \qquad 0 < a < r < b, \quad -\pi < \theta < \pi, \tag{1a}$$

$$\frac{\partial u}{\partial r}(a,\theta) = 0, \quad u(b,\theta) = f(\theta), \qquad -\pi < \theta < \pi, \tag{1b}$$

$\boxed{2}$ (a) Sketch the region on which the PDE problem (1) is defined, and label the boundaries according to (1b).




$\boxed{4}$ (b) Set up *(do not solve!)* the two ODE problems that arise from using the separation of variables technique to solve the PDE problem. *Don't forget to specify the boundary conditions for one of the ODE problems!*

Let $u(r,\theta) = R(r)T(\theta)$. Then (1a) becomes

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0 \iff r^2R''T + rR'T + RT'' = 0$$

$$\iff \frac{r^2 R''T}{RT} + \frac{rR'T}{RT} + \frac{RT''}{RT} = 0$$

$$\iff \frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{T''}{T} = \lambda$$

*extra workspace for question 1*

We thus arrive at the two ODE problems

$$\text{(A)} \begin{cases} T'' + \lambda T = 0 \\ T(-\pi) = T(\pi) \\ T'(-\pi) = T'(\pi) \end{cases} \qquad \text{(B)} \begin{cases} r^2 R'' + r R' - \lambda R = 0 \\ \end{cases}$$

2. The Laplace equation on a disc of radius $r = 4$ leads to two ODE problems: The first is a BVP in $\theta$,

$$T''(\theta) + \alpha^2 T(\theta) = 0, \quad -\pi < \theta < \pi, \tag{2a}$$
$$T(-\pi) = T(\pi), \quad T'(-\pi) = T'(\pi), \tag{2b}$$

and the second is a Cauchy-Euler ODE in $r$,

$$r^2 R''(r) + rR'(r) - \alpha^2 R(r) = 0, \quad 0 < r < 4. \tag{3}$$

Assume $\alpha > 0$.

$\boxed{5}$ (a) Solve the BVP (2).

The solutions of (2a) are

$T(\theta) = c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta)$

and so we have

$T'(\theta) = -\alpha c_1 \sin(\alpha\theta) + \alpha c_2 \cos(\alpha\theta)$

Applying the BCs we obtain:

$$\begin{cases} T(-\pi) = T(\pi) \\ T'(-\pi) = T'(\pi) \end{cases} \Leftrightarrow \begin{cases} \cancel{c_1\cos(\alpha\pi)} - c_2\sin(\alpha\pi) = \cancel{c_1\cos(\alpha\pi)} + c_2\sin(\alpha\pi) \\ \alpha c_1\sin(\alpha\pi) + \cancel{\alpha c_2\cos(\alpha\pi)} = -\alpha c_1\sin(\alpha\pi) + \cancel{\alpha c_2\cos(\alpha\pi)} \end{cases}$$

$$\Leftrightarrow \begin{cases} 2c_2\sin(\alpha\pi) = 0 \\ 2\alpha c_1 \sin(\alpha\pi) = 0 \end{cases}$$

For nontrivial solutions, we require

$\sin(\alpha\pi) = 0 \Leftrightarrow \alpha\pi = n\pi \Leftrightarrow \alpha = n, \; n \in \mathbb{N}$

and

$T_n(\theta) = c_{1n}\cos(n\theta) + c_{2n}\sin(n\theta)$

4 (b) Solve the ODE (3). *Note the domain!!*

Since $\alpha = n$, $n \in \mathbb{N}$, we have

$$r^2 R_n'' + r R_n' - n^2 R = 0$$

The solutions of this Cauchy-Euler equation are of the form $R_n(r) = r^p$. Then

$$r^2 p(p-1) r^{p-2} + r p r^{p-1} - n^2 r^p = 0 \iff r^p (p^2 - p + p - n^2) = 0$$
$$\iff r^p (p^2 - n^2) = 0$$

$\because$ $r^p \neq 0$ we must have $p^2 = n^2 \iff p = \pm n$

$\therefore$ $R_n(r) = d_{1n} r^n + d_{2n} r^{-n}$

Since the solution domain includes points arbitrarily close to $r = 0$, we must have $d_{2n} = 0$ in order to have finite solutions. So

$$R_n(r) = d_{1n} r^n$$

3. Consider the PDE problem

$$\frac{\partial^2 u}{\partial t^2} = \beta \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2\pi, \quad t > 0, \tag{4a}$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(2\pi, t) = 0, \qquad t > 0, \tag{4b}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < 2\pi, \tag{4c}$$

where $\beta > 0$. By assuming $u(x,t) = X(x)T(t)$, this PDE becomes two ODE problems:

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(2\pi) = 0 \tag{5}$$

$$T'' + \lambda \beta T = 0 \tag{6}$$

$\boxed{1}$ (a) Why do the two initial conditions not appear in the ODE problem for $T(t)$?

The two ICs are nonhomogeneous, so we can't consider them at the ODE stage.

$\boxed{7}$ (b) Assuming that $\lambda = w^2$, find a formal solution for $u(x,t)$.

We first solve (5):

$X'' + w^2 X = 0 \iff X(x) = c_1 \cos(wx) + c_2 \sin(wx)$

Apply the BCs:

$X'(0) = 0 \iff c_2 = 0$ (since $\sin(wx)$ does not satisfy this condition)

$X(2\pi) = 0 \iff c_1 \cos(2\pi w) = 0$

For nontrivial solutions, we require

$2\pi w = (2n-1)\frac{\pi}{2}, \; n \in \mathbb{N} \iff w = \frac{(2n-1)}{4}, \; n \in \mathbb{N}$

and $X_n(x) = \cos\left(\frac{(2n-1)}{4} x\right), \; n \in \mathbb{N}$

*extra workspace for question 3*

Now consider (ii). We have

$$T'' + \beta\omega^2 T = 0 \iff T(t) = c_1\cos(\omega\sqrt{\beta}\,t) + c_2\sin(\omega\sqrt{\beta}\,t), \ \because \beta>0$$

where $\omega$ is given above.

Now form $u$:

$$u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} k_n X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} \cos\left(\frac{(2n-1)}{4}x\right)\left[A_n\cos\left(\sqrt{\beta}\,\frac{(2n-1)}{4}t\right) + B_n\sin\left(\sqrt{\beta}\,\frac{(2n-1)}{4}t\right)\right] \quad \ldots (*)$$

Now apply the ICs:

$$u(x,0) = f(x) \iff \sum_{n=1}^{\infty} A_n\cos\left(\frac{(2n-1)}{4}x\right) = f(x)$$

$$\text{So } A_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{(2n-1)}{4}x\right)dx \quad \ldots (**)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \iff \sum_{n=1}^{\infty} B_n\sqrt{\beta}\,\frac{(2n-1)}{4}\cos\left(\frac{(2n-1)}{4}x\right) = g(x)$$

$$\text{So } B_n\sqrt{\beta}\,\frac{(2n-1)}{4} = \frac{2}{2\pi}\int_0^{2\pi} g(x)\cos\left(\frac{(2n-1)}{4}x\right)dx \quad \ldots (***)$$

Together, (*), (**), and (***) give the formal solution to (4).

[6] 4. Use Separation of Variables to reduce the PDE problem below to three ODE problems (do not solve the ODE problems, just set them up).

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x < a, \quad 0 < y < b, \quad t > 0, \tag{7a}$$

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(a, y, t) = 0, \qquad 0 < y < b, \quad t > 0, \tag{7b}$$

$$u(x, 0, t) = U_1, \quad u(x, b, t) = U_2, \qquad 0 < x < a, \quad t > 0, \tag{7c}$$

$$u(x, y, 0) = g(x, y), \qquad 0 < x < a, \quad 0 < y < b. \tag{7d}$$

Include boundary conditions in the ODE problems where appropriate.

Let $u(x,y,t) = X(x)Y(y)T(t)$. Then (7a) becomes

$$XYT' = X''YT + XY''T \iff \frac{XYT'}{XYT} = \frac{X''YT}{XYT} + \frac{XY''T}{XYT}$$

$$\iff \frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} \quad \ldots (8)$$

$\because$ the BCs in $x$ are homogeneous, we will use them to define our initial BVP, & so we rearrange (8):

$$\frac{X''}{X} = \frac{T'}{T} - \frac{Y''}{Y} = -\lambda \quad \ldots (9)$$

So the first ODE problem is

$$X'' + \lambda X = 0, \quad X'(0) = X'(a) = 0 \quad \ldots (10)$$

We now take the RHS of (9) & obtain

$$\frac{T'}{T} - \frac{Y''}{Y} = -\lambda \iff \frac{T'}{T} + \lambda = \frac{Y''}{Y} = -\mu \quad \ldots (11)$$

From (11) we have two more ODEs:

$$Y'' + \mu Y = 0 \quad \ldots (12)$$

$$T' + (\lambda + \mu)T = 0 \quad \ldots (13)$$

Eqns (10), (12), & (13) are the three ODE problems arising fr (7).