More about ODE - To solve PDEs, we will often (almost always!) transform the PDE into ODEs, & solve those. So let's review...

## Simple Examples

### $1^{st}$ order linear homogeneous ODE

$\boxed{Ex1}$ $y' - y = 0$ ($y' + e^y = 0$ is NOT linear)

SOL'N: look for $y$ s.t. $y' = y$

- guess: $y(t) = e^t$, $y'(t) = e^t$
  $y(t) = Ae^t$, $y'(t) = \boxed{Ae^t}$

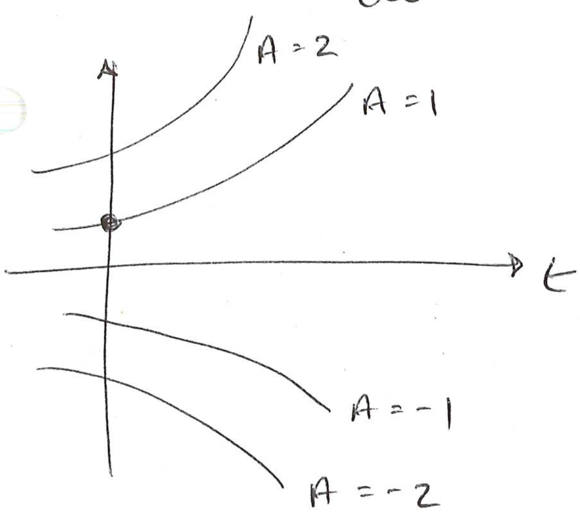
- Separation of variables:

$$\frac{dy}{dt} - y = 0 \iff \frac{dy}{dt} = y \iff \int \frac{dy}{y} = \int dt$$

$$\iff \ln_e(y) = t + c_e$$

$$\iff y = e^{t+c} = \boxed{Ae^t}$$

family of solutions (general solution)




Each solution is determined by imposing an initial condition $y(0) =$ ___ .

### $2^{nd}$ order linear homogeneous ODE

$\boxed{Ex2}$ $y'' - y' - 2y = 0$

Try $y = e^{rt}$, then we obtain

$y' = re^{rt}$

$y'' = r^2e^{rt}$

Plugging into the ODE:

$$r^2e^{rt} - re^{rt} - 2e^{rt} = 0 \iff r^2 - r - 2 = 0$$
$$\iff (r-2)(r+1) = 0$$
$$\iff r = 2 \text{ or } r = -1$$

$\therefore$ we have two solutions $y_1 = e^{2t}$, $y_2 = e^{-t}$, and the <u>general</u> solution is

$$y = c_1 e^{2t} + c_2 e^{-t}$$

two-parameter family of solutions

Each solution is determined by imposing TWO initial conditions:

$$y(t_0) = \_\_\_\_ , \quad y'(t_0) = \_\_\_\_ .$$

$\boxed{\text{Ex 3}}$ <u>Another important ODE (2nd order, linear)</u>

$$y'' + y = 0 \iff r^2 + 1 = 0 \iff r^2 = -1 \iff r = \pm i$$

$$\therefore y_1(t) = e^{it}, \quad y_2(t) = e^{-it}$$

$$\boxed{\text{Euler's Formula: } e^{it} = \cos(t) + i\sin(t)}$$

We find that the real & imaginary parts form a <u>fundamental solution set</u> for the ODE, and so

$$y(t) = c_1\cos(t) + c_2\sin(t)$$

# PDE : A simple Example

Ex. 1. $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0$

1st-order PDE
linear, constant coefficient
Homogeneous.

$u = u(x,t)$

(*) guess Consider $u(x,t) = F(x-t)$

$$\frac{\partial u}{\partial t} = -F'(x-t)$$

$$\frac{\partial u}{\partial x} = F'(x-t)$$

Subst. back in P.D.E

$$-F(x-t) + F(x-t) = 0 \quad \checkmark$$

* solution is any arbitrary function of $x-t$

i.e. $(x-t)^2$, $\sin(x-t)$, $e^{x-t}$, etc...

* To choose a particular solution, we need to specify $u$ along some curve in the $x,t$-plane.

$$u(x,0) = F(x)$$

Say $u(x,0) = e^{-x^2} = F(x) \Rightarrow$ solution. $u(x,t) = F(x-t)$

$$= e^{-(x-t)^2}$$

look at section.

* Could have

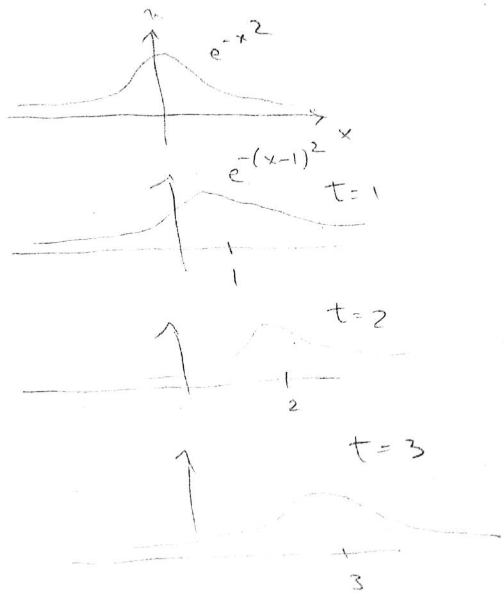
$$u(x,t) = A e^{-(x-t)^2}$$

at $t=0$

$t=1$

$t=2$

$t=3$




solution is a <u>surface</u>


* a wave form moving to the right as t increase without changing its shape.

recall

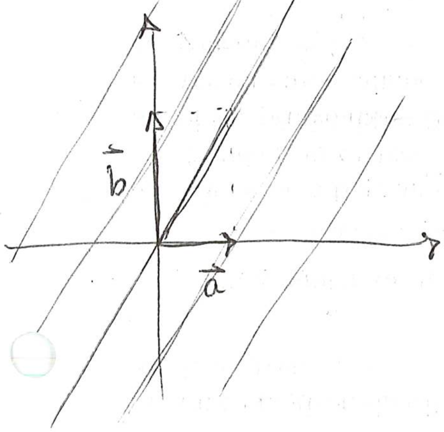
| |
|---|
| ODE |
| one degree of freedom ⇒ specify one point <br> 0-dim |
| PDE & constant |
| one degree of freedom ⇒ specify a curve <br> 1-dim. |

Consider the general constant-coefficient case:

$$\boxed{a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}} + cu = f(x,y)$$

non-hom, linear, cst coeff

$$a u_x + b u_y = (a,b)\cdot(u_x,u_y)$$
$$= \underbrace{(a\hat{i} + b\hat{j})\cdot\vec{\nabla} u}$$




This is the directional derivative of $u$ in the direction of the vector $(a,b)$.

So we consider a family of lines w/ slope $b/a$:

$bx - ay = d$ : lines // to $a\hat{i} + b\hat{j}$

These are the $\boxed{\text{characteristic lines}}$ of the PDE.

Choose new coordinates,

$$\begin{cases} w = bx - ay, \\ z = y, \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{b}(w + az), \\ y = z, \end{cases}$$

and set $v(w,z) = u(x,y)$. Then

$$a u_x + b u_y = a(v_w w_x + v_z z_x) + b(v_w w_y + v_z z_y)$$
$$= abv_w + b((-a)v_w + v_z)$$
$$= bv_z$$

and the PDE simplifies to

$$\boxed{bv_z + cv = f\left(\frac{1}{b}(w+az), z\right).}$$

Ex (if time) (next p)

# Math 319
# Partial Differential Equations
# Pre-Lecture Assignment #2
# due ~~Thu~~ Tu September ~~8th~~ 13th, 10am

**Instructions:** Your work will be marked before class and the solution will be discussed as a group during the first few minutes of the lecture.

1. **Review of ODEs:** Solve the IVP

$$y'' + 2y' + 3y = 0, \qquad y(0) = 0, y'(0) = 1.$$

2. **Method of Characteristics:** Consider the PDE

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + 5u = \sin(xy). \tag{1}$$

   (a) Find the preferred direction and family of characteristic lines all parallel to the preferred direction vector.

   (b) Write the equations for the new variables $w$ and $z$ in terms of $x$ and $y$.

   (c) Rewrite the PDE (1) in terms of the new variables.

   (d) Form the integrating factor

$$\mu(z) = exp\left(\int \frac{c}{b}dz\right).$$

~~3. Do you have any questions about the syllabus or other aspect of the course?~~

# Pre-Reading #2 - Sol'ns

1. $y'' + 2y' + 3y = 0$ $\qquad y(0) = 0, \; y'(0) = 1.$

char eqn: $r^2 + 2r + 3 = 0 \iff r = -1 \pm \sqrt{1-3} = -1 \pm \sqrt{-2} = -1 \pm \sqrt{2}\,i$

$\therefore \; y(t) = e^{-t}\left(c_1 \cos(\sqrt{2}\,t) + c_2 \sin(\sqrt{2}\,t)\right)$

apply ICs:

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \iff \begin{cases} c_1 = 0 \\ \left[ -e^{-t} \sin(\sqrt{2}\,t)\Big|_{t=0} + e^{-t}\sqrt{2}\cos(\sqrt{2}\,t)\Big|_{t=0} \right] c_2 = 1 \end{cases}$$

$$\iff \begin{cases} c_1 = 0 \\ \sqrt{2}\, c_2 = 1 \end{cases} \iff \begin{cases} c_1 = 0 \\ c_2 = 1/\sqrt{2} \end{cases}$$

$\therefore \; y(t) = \dfrac{e^{-t}}{\sqrt{2}} \sin(\sqrt{2}\,t)$ $\qquad$ damped oscillations (underdamped)

2. $3u_x + 2u_y + 5u = \sin(xy)$

a) preferred direction $m = \frac{2}{3}$ or $\vec{m} = (3\hat{i} + 2\hat{j}) = (3, 2)$

b) $\begin{cases} w = bx - ay \\ z = y \end{cases} \iff \begin{cases} w = 2x - 3y \\ z = y \end{cases} \iff \begin{cases} x = \frac{1}{2}(w + 3z) \\ y = z \end{cases}$

c) let $v(w, z) = u(x, y)$. Then

$3u_x + 2u_y = 2v_z \Rightarrow 2v_z + 5v = \sin\left(\frac{1}{2}(w + 3z) \cdot z\right)$

d) $\mu(z) = e^{\int \frac{5}{2} dz} = e^{\frac{5}{2} z}$ (ignore the cst)

For lecture: Finish the solution

(for PL#2)

$$2v_z + 5v = \sin\left(\frac{(w+3z)\cdot z}{2}\right) \iff 1/$$

$$1/ \iff \frac{dv}{dz} + \frac{5}{2}v = \frac{1}{2}\sin\left(\frac{(w+3z)z}{2}\right)$$

$$\iff e^{\frac{5}{2}z}\frac{dv}{dz} + e^{\frac{5}{2}z}v = \frac{1}{2}e^{\frac{5}{2}z}\sin\left(\frac{(w+3z)z}{2}\right)$$

$$\iff \frac{d}{dz}\left(e^{\frac{5}{2}z}v\right) = \frac{1}{2}e^{\frac{5}{2}z}\sin\left(\frac{(w+3z)\cdot z}{2}\right)$$

$$\iff e^{\frac{5}{2}z}v = \int \frac{1}{2}e^{\frac{5}{2}z}\sin\left(\frac{(w+3z)\cdot z}{2}\right)dz + C(w)$$

$$\iff v = e^{-\frac{5}{2}z}\int \frac{1}{2}e^{\frac{5}{2}z}\sin\left(\frac{(w+3z)\cdot z}{2}\right)dz + e^{-\frac{5}{2}z}C(w)$$

~~To determine C(w) need IC~~

In terms of $x$ and $y$ we have

$$u(x,y) = e^{-\frac{5}{2}y} \int \frac{1}{2} e^{\frac{5}{2}y} \sin( xy ) \, dy$$

$$+ \; C(2x-3y)$$

$$= \frac{e^{-\frac{5}{2}y}}{2} \int e^{\frac{5}{2}y} \sin( xy ) \, dy + C(2x-3y)$$

In order to determine $C$, we require $u(x,0)$ or $u(0,y)$.